

Common Math Errors

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Originally the intended audience for this was my Calculus I students as pretty much every error listed here shows up in that class with alarming frequency. After writing it however I realized that, with the exception of a few examples, the first four sections should be accessible to anyone taking a math class and many of the errors listed in the first four sections also show up in math classes at pretty much every level. So, if you haven't had calculus yet (or never will) you should ignore the last section and the occasional calculus examples in the first four sections.

I got the idea for doing this when I ran across Eric Schechter's list of common errors located at <http://www.math.vanderbilt.edu/~schectex/commerrs/>. There is a fair amount of overlap in the errors discussed on both of our pages. Sometimes the discussion is similar and at other times it's different. The main difference between our two pages is I stick to the level of Calculus and lower while he also discusses errors in proof techniques and some more advanced topics as well. I would encourage everyone interested in common math errors also take a look at his page.

General Errors

In fact, that was the original title of this section, but I did not want to leave people with the feeling that I'm trying to imply that math is easy and that everyone should just "get it"! For many people math is a very difficult subject and they will struggle with it. So please do not leave with the impression that I'm trying to imply that math is easy for everyone. The intent of this section is to address certain attitudes and preconceptions many students have that can make a math class very difficult to successfully complete.

Putting off math requirements

I don't know how many students have come up to me and said something along the lines of:

"I've been putting this off for a while now because math is so hard for me and now I've got to have it in order to graduate this semester."

This has got to be one of the strangest attitudes that I've ever run across. If math is hard for you, putting off your math requirements is one of the worst things that you can do! You should take your math requirements as soon as you can. There are several reasons for this.

The first reason can be stated in the following way : MATH IS CUMULATIVE. In other words, most math classes build on knowledge you've gotten in previous math classes, including your high school math classes. So, the only real effect of putting off your math requirement is that you forget the knowledge that you once had. It will be assumed that you've still got this knowledge when you finally do take your math requirement!

If you put off your math requirement you will be faced with the unpleasant situation of having to learn new material AND relearn all the forgotten material at the same time. In most cases, this means that you will struggle in the class far more than if you had just taken it right away!

The second reason has nothing to do with knowledge (or the loss of knowledge), but instead has everything to do with reality. If math is hard for you and you struggle to pass a math course, then you really should take the course at a time that allows for the unfortunate possibility that you don't pass. In other words, to put it bluntly, if you wait until your last semester to take your required math course and fail you won't be graduating! Take it right away so if you do unfortunately fail the course you can retake it the next semester.

This leads to the third reason. Too many students wait until the last semester to take their math class in the hopes that their instructor will take pity on them and not fail them because they're graduating. To be honest the only thing that I, and many other instructors, feel in these cases is irritation at being put into the position at having to be the

bad guy and failing a graduating senior. Not a situation where you can expect much in the way of sympathy!

Doing the bare minimum

I see far too many students trying to do the bare minimum required to pass the class, or at least what they feel is the bare minimum required. The problem with this is they often underestimate the amount of work required early in the class, get behind, and then spend the rest of the semester playing catch up and having to do far more than just the bare minimum.

You should always try to get the best grade possible! You might be surprised and do better than you expected. At the very least you will lessen the chances of underestimating the amount of work required and getting behind.

Remember that math is NOT a spectator sport! You must be actively involved in the learning process if you want to do well in the class.

A good/bad first exam score doesn't translate into a course grade

Another heading here could be : "Don't get cocky and don't despair". If you get a good score on the first exam do not decide that means that you don't need to work hard for the rest of the semester. All the good score means is that you're doing the proper amount of for studying for the class! Almost every semester I have a student get an A on the first class and end up with a C (or less) for the class because he/she got cocky and decided to not study as much and promptly started getting behind and doing poorly on exams.

Likewise, if you get a bad score on the first exam do not despair! All the bad score means is that you need to do a little more work for the next exam. Work more problems, join a study group, or get a tutor to help you. Just as I have someone go downhill almost every semester I also have at least one student who fails the first exam and yet passes the class, often with a B and occasionally an A!

Your score on the first exam simply doesn't translate into a course grade. There is a whole semester in front of you and lot's of opportunities to improve your grade so don't despair if you didn't do as well as you wanted to on the first exam.

Expecting to instantly understand a concept/topic/section

Assuming that if it's "easy" in class it will be "easy" on the exam

Don't know how to study mathematics

The first two are really problems that fall under the last topic but I run across them often enough that I thought I'd go ahead and put them down as well. The reality is that most

people simply don't know how to study mathematics. This is not because people are not capable of studying math, but because they've never really learned how to study math.

Mathematics is not like most subjects and accordingly you must also study math differently. This is an unfortunate reality and many students try to study for a math class in the same way that they would study for a history class, for example. This will inevitably lead to problems. In a history class you can, in many cases, simply attend class memorize a few names and/or dates and pass the class. In a math class things are different. Simply memorizing will not always get you through the class, you also need to understand HOW to use the formula that you've memorized.

This is such an important topic and there is so much to be said I've devoted a whole document to just this topic. My **How To Study Mathematics** can be accessed at,

<http://tutorial.math.lamar.edu/Extras/StudyMath/HowToStudyMath.aspx>

Algebra Errors

The topics covered here are errors that students often make in doing algebra, and not just errors typically made in an algebra class. I've seen every one of these mistakes made by students in all level of classes, from algebra classes up to senior level math classes! In fact a few of the examples in this section will actually come from calculus.

If you have not had calculus you can ignore these examples. In every case were I've given examples I've tried to include examples from an algebra class as well as the occasion example from upper level courses like Calculus.

I'm convinced that many of these are mistakes given here are caused by people getting lazy or getting in a hurry and not paying attention to what they're doing. By slowing down, paying attention to what you're doing and paying attention to proper notation you can avoid the vast majority of these mistakes!

Division by Zero

Everyone knows that $\frac{0}{2} = 0$ the problem is that far too many people also say that $\frac{2}{0} = 0$ or $\frac{2}{0} = 2$! Remember that division by zero is undefined! You simply cannot divide by zero so don't do it!

Here is a very good example of the kinds of havoc that can arise when you divide by zero. See if you can find the mistake that I made in the work below.

- | | |
|--------------------------------|--|
| 1. $a = b$ | We'll start assuming this to be true. |
| 2. $ab = a^2$ | Multiply both sides by a . |
| 3. $ab - b^2 = a^2 - b^2$ | Subtract b^2 from both sides. |
| 4. $b(a - b) = (a + b)(a - b)$ | Factor both sides. |
| 5. $b = a + b$ | Divide both sides by $a - b$. |
| 6. $b = 2b$ | Recall we started off assuming $a = b$. |
| 7. $1 = 2$ | Divide both sides by b . |

So, we've managed to prove that $1 = 2$! Now, we know that's not true so clearly we made a mistake somewhere. Can you see where the mistake was made?

The mistake was in step 5. Recall that we started out with the assumption $a = b$. However, if this is true then we have $a - b = 0$! So, in step 5 we are really dividing by zero!

That simple mistake led us to something that we knew wasn't true, however, in most cases your answer will not obviously be wrong. It will not always be clear that you are dividing by zero, as was the case in this example. You need to be on the lookout for this kind of thing.

Remember that you CAN'T divide by zero!

Bad/lost/Assumed Parenthesis

This is probably error that I find to be the most frustrating. There are a couple of errors that people commonly make here.

The first error is that people get lazy and decide that parenthesis aren't needed at certain steps or that they can remember that the parenthesis are supposed to be there. Of course the problem here is that they often tend to forget about them in the very next step!

The other error is that students sometimes don't understand just how important parentheses really are. This is often seen in errors made in exponentiation as my first couple of examples show.

Example 1 Square $4x$.

Correct	Incorrect
$(4x)^2 = (4)^2(x)^2 = 16x^2$	$4x^2$

Note the very important difference between these two! When dealing with exponents remember that only the quantity immediately to the left of the exponent gets the exponent. So, in the incorrect case, the x is the quantity immediately to the left of the exponent so we are squaring only the x while the 4 isn't squared. In the correct case the parenthesis is immediately to the left of the exponent so this signifies that everything inside the parenthesis should be squared!

Parenthesis are required in this case to make sure we square the whole thing, not just the x , so don't forget them!

Example 2 Square -3 .

Correct	Incorrect
$(-3)^2 = (-3)(-3) = 9$	$-3^2 = -(3)(3) = -9$

This one is similar to the previous one, but has a subtlety that causes problems on

occasion. Remember that only the quantity to the left of the exponent gets the exponent. So, in the incorrect case ONLY the 3 is to the left of the exponent and so ONLY the 3 gets squared!

Many people know that technically they are supposed to square -3, but they get lazy and don't write the parenthesis down on the premise that they will remember them when the time comes to actually evaluate it. However, it's amazing how many of these folks promptly forget about the parenthesis and write down -9 anyway!

Example 3 Subtract $4x - 5$ from $x^2 + 3x - 5$

Correct

$$\begin{aligned} x^2 + 3x - 5 - (4x - 5) &= x^2 + 3x - 5 - 4x + 5 \\ &= x^2 - x \end{aligned}$$

Incorrect

$$x^2 + 3x - 5 - 4x - 5 = x^2 - x - 10$$

Be careful and note the difference between the two! In the first case I put parenthesis around then $4x - 5$ and in the second I didn't. Since we are subtracting a polynomial we need to make sure we subtract the WHOLE polynomial! The only way to make sure we do that correctly is to put parenthesis around it.

Again, this is one of those errors that people do know that technically the parenthesis should be there, but they don't put them in and promptly forget that they were there and do the subtraction incorrectly.

Example 4 Convert $\sqrt{7x}$ to fractional exponents.

Correct

$$\sqrt{7x} = (7x)^{\frac{1}{2}}$$

Incorrect

$$\sqrt{7x} = 7x^{\frac{1}{2}}$$

This comes back to same mistake in the first two. If only the quantity to the left of the exponent gets the exponent. So, the incorrect case is really $7x^{\frac{1}{2}} = 7\sqrt{x}$ and this is clearly NOT the original root.

Example 5 Evaluate $-3\int 6x - 2 dx$.

This is a calculus problem, so if you haven't had calculus you can ignore this example. However, far too many of my calculus students make this mistake for me to ignore it.

Correct

$$\begin{aligned} -3\int 6x - 2 dx &= -3(3x^2 - 2x) + c \\ &= -9x^2 + 6x + c \end{aligned}$$

Incorrect

$$\begin{aligned} -3\int 6x - 2 dx &= -3 \cdot 3x^2 - 2x + c \\ &= -9x^2 - 2x + c \end{aligned}$$

Note the use of the parenthesis. The problem states that it is -3 times the **WHOLE** integral not just the first term of the integral (as is done in the incorrect example).

Improper Distribution

Be careful when using the distribution property! There two main errors that I run across on a regular basis.

Example 1 Multiply $4(2x^2 - 10)$.

Correct	Incorrect
$4(2x^2 - 10) = 8x^2 - 40$	$4(2x^2 - 10) = 8x^2 - 10$

Make sure that you distribute the 4 all the way through the parenthesis! Too often people just multiply the first term by the 4 and ignore the second term. This is especially true when the second term is just a number. For some reason, if the second term contains variables students will remember to do the distribution correctly more often than not.

Example 2 Multiply $3(2x - 5)^2$.

Correct	Incorrect
$3(2x - 5)^2 = 3(4x^2 - 20x + 25)$	$3(2x - 5)^2 = (6x - 15)^2$
$= 12x^2 - 60x + 75$	$= 36x^2 - 180x + 225$

Remember that exponentiation must be performed **BEFORE** you distribute any coefficients through the parenthesis!

Additive Assumptions

I didn't know what else to call this, but it's an error that many students make. Here's the assumption. Since $2(x + y) = 2x + 2y$ then everything works like this. However, here is a whole list in which this doesn't work.

$$(x + y)^2 \neq x^2 + y^2$$

$$\sqrt{x + y} \neq \sqrt{x} + \sqrt{y}$$

$$\frac{1}{x + y} \neq \frac{1}{x} + \frac{1}{y}$$

$$\cos(x + y) \neq \cos x + \cos y$$

It's not hard to convince yourself that any of these aren't true. Just pick a couple of numbers and plug them in! For instance,

$$(1+3)^2 \neq 1^2 + 3^2$$

$$(4)^2 \neq 1+9$$

$$16 \neq 10$$

You will find the occasional set of numbers for which one of these rules will work, but they don't work for almost any randomly chosen pair of numbers.

Note that there are far more examples where this *additive* assumption doesn't work than what I've listed here. I simply wrote down the ones that I see most often. Also a couple of those that I listed could be made more general. For instance,

$$(x+y)^n \neq x^n + y^n \quad \text{for any integer } n \geq 2$$

$$\sqrt[n]{x+y} \neq \sqrt[n]{x} + \sqrt[n]{y} \quad \text{for any integer } n \geq 2$$

Canceling Errors

These errors fall into two categories. Simplifying rational expressions and solving equations. Let's look at simplifying rational expressions first.

Example 1 Simplify $\frac{3x^3 - x}{x}$ (done correctly).

$$\frac{3x^3 - x}{x} = \frac{x(3x^2 - 1)}{x} = 3x^2 - 1$$

Notice that in order to cancel the x out of the denominator I first factored an x out of the numerator. You can only cancel something if it is multiplied by the WHOLE numerator and denominator, or if IS the whole numerator or denominator (as in the case of the denominator in our example).

Contrast this with the next example which contains a very common error that students make.

Example 2 Simplify $\frac{3x^3 - x}{x}$ (done incorrectly).

Far too many students try to simplify this as,

$$3x^2 - x \quad \text{OR} \quad 3x^3 - 1$$

In other words, they cancel the x in the denominator against only one of the x 's in the

numerator (*i.e.* cancel the x only from the first term or only from the second term). THIS CAN'T BE DONE!!!! In order to do this canceling you MUST have an x in both terms.

To convince yourself that this kind of canceling isn't true consider the following number example.

Example 3 Simplify $\frac{8-3}{2}$.

This can easily be done just by doing the arithmetic as follows

$$\frac{8-3}{2} = \frac{5}{2} = 2.5$$

However, let's do an incorrect cancel similar to the previous example. We'll first cancel the two in the denominator into the eight in the numerator. This is NOT CORRECT, but it mirrors the canceling that was incorrectly done in the previous example. This gives,

$$\frac{8-3}{2} = 4-3 = 1$$

Clearly these two aren't the same! So you need to be careful with canceling!

Now, let's take a quick look at canceling errors involved in solving equations.

Example 4 Solve $2x^2 = x$ (done incorrectly).

Too many students get used to just canceling (*i.e.* simplifying) things to make their life easier. So, the biggest mistake in solving this kind of equation is to cancel an x from both sides to get,

$$2x = 1 \quad \Rightarrow \quad x = \frac{1}{2}$$

While, $x = \frac{1}{2}$ is a solution, there is another solution that we've missed. Can you see what it is? Take a look at the next example to see what it is.

Example 5 Solve $2x^2 = x$ (done correctly).

Here's the correct way to solve this equation. First get everything on one side then factor!

$$2x^2 - x = 0$$

$$x(2x - 1) = 0$$

From this we can see that either

$$x = 0 \quad \text{OR} \quad 2x - 1 = 0$$

In the second case we get the $x = \frac{1}{2}$ we got in the first attempt, but from the first case we also get $x = 0$ that we didn't get in the first attempt. Clearly $x = 0$ will work in the equation and so is a solution!

We missed the $x = 0$ in the first attempt because we tried to make our life easier by “simplifying” the equation before solving. While some simplification is a good and necessary thing, you should NEVER divide out a term as we did in the first attempt when solving. If you do this you WILL lose solutions.

Proper Use of Square Root

There seems to be a very large misconception about the use of square roots out there. Students seem to be under the misconception that

$$\sqrt{16} = \pm 4$$

This is not correct however. Square roots are ALWAYS positive! So the correct value is

$$\sqrt{16} = 4$$

This is the ONLY value of the square root! If we want the -4 then we do the following

$$-\sqrt{16} = -(\sqrt{16}) = -(4) = -4$$

Notice that I used parenthesis only to make the point on just how the minus sign was appearing! In general the middle two steps are omitted. So, if we want the negative value we have to actually put in the minus sign!

I suppose that this misconception arises because they are also asked to solve things like $x^2 = 16$. Clearly the answer to this is $x = \pm 4$ and often they will solve by “taking the square root” of both sides. There is a missing step however. Here is the proper solution technique for this problem.

$$x^2 = 16$$

$$x = \pm\sqrt{16}$$

$$x = \pm 4$$

Note that the \pm shows up in the second step before we actually find the value of the square root! It doesn't show up as part of taking the square root.

I feel that I need to point out that many instructors (including myself on occasion) don't help matters in that they will often omit the second step and by doing so seem to imply that the \pm is showing up because of the square root.

So remember that square roots ALWAYS return a positive answer. If you want a negative you'll need to put it in a minus sign BEFORE you take the square root.

Ambiguous Fractions

This is more a notational issue than an algebra issue. I decided to put it here because too many students come out of algebra classes without understanding this point. There are really three kinds of "bad" notation that people often use with fractions that can lead to errors in work.

The first is using a "/" to denote a fraction, for instance $2/3$. In this case there really isn't a problem with using a "/", but what about $2/3x$? This can be either of the two following fractions.

$$\frac{2}{3}x \quad \text{OR} \quad \frac{2}{3x}$$

It is not clear from $2/3x$ which of these two it should be! You, as the student, may know which one of the two that you intended it to be, but a grader won't. Also, while you may know which of the two you intended it to be when you wrote it down, will you still know which of the two it is when you go back to look at the problem when you study?

You should only use a "/" for fractions when it will be clear and obvious to everyone, not just you, how the fraction should be interpreted.

The next notational problem I see fairly regularly is people writing $\frac{2}{3}x$. It is not clear from this if the x belongs in the denominator or the fraction or not. Students often write fractions like this and usually they mean that the x shouldn't be in the denominator. The problem is on a quick glance it often looks like it should be in the denominator and the student just didn't draw the fraction bar over far enough.

If you intend for the x to be in the denominator then write it as such that way, $\frac{2}{3x}$, *i.e.* make sure that you draw the fraction bar over the WHOLE denominator. If you don't intend for it to be in the denominator then don't leave any doubt! Write it as $\frac{2}{3}x$.

The final notational problem that I see comes back to using a “/” to denote a fraction, but is really a parenthesis problem. This involves fractions like

$$\frac{a+b}{c+d}$$

Often students who use “/” to denote fractions will write this is fraction as

$$a+b/c+d$$

These students know that they are writing down the original fraction. However, almost anyone else will see the following

$$a+\frac{b}{c}+d$$

This is definitely NOT the original fraction. So, if you MUST use “/” to denote fractions use parenthesis to make it clear what is the numerator and what is the denominator. So, you should write it as

$$(a+b)/(c+d)$$

Trig Errors

This is a fairly short section, but contains some errors that I see my calculus students continually making so I thought I'd include them here as a separate section.

Degrees vs. Radians

Most trig classes that I've seen taught tend to concentrate on doing things in degrees. I suppose that this is because it's easier for the students to visualize, but the reality is that almost all of calculus is done in radians and students too often come out of a trig class ill prepared to deal with all the radians in a calculus class.

You simply must get used to doing everything in radians in a calculus class. If you are asked to evaluate $\cos(x)$ at $x=10$ we are asking you to use 10 radians not 10 degrees!

The answers are very, very different! Consider the following,

$$\cos(10) = -0.839071529076 \quad \text{in radians}$$

$$\cos(10) = 0.984807753012 \quad \text{in degrees}$$

You'll notice that they aren't even the same sign!

So, be careful and make sure that you always use radians when dealing with trig functions in a trig class. Make sure your calculator is set to calculations in radians.

cos(x) is NOT multiplication

I see students attempting both of the following on a continual basis

$$\cos(x + y) \neq \cos(x) + \cos(y)$$

$$\cos(3x) \neq 3 \cos(x)$$

These just simply aren't true. The only reason that I can think of for these mistakes is that students must be thinking of $\cos(x)$ as a multiplication of something called **cos** and x . This couldn't be farther from the truth! Cosine is a function and the **cos** is used to denote that we are dealing with the cosine function!

If you're not sure you believe that those aren't true just pick a couple of values for x and y and plug into the first example.

$$\cos(\pi + 2\pi) \neq \cos(\pi) + \cos(2\pi)$$

$$\cos(3\pi) \neq -1 + 1$$

$$-1 \neq 0$$

So, it's clear that the first isn't true and we could do a similar test for the second example.

$$\begin{aligned}\cos(3\pi) &\neq 3\cos(\pi) \\ -1 &\neq 3(-1) \\ -1 &\neq -3\end{aligned}$$

I suppose that the problem is that occasionally there are values for these that are true. For instance, you could use $x = \frac{\pi}{2}$ in the second example and both sides would be zero and so it would work for that value of x . In general however, for the vast majority of values out there in the world these simply aren't true!

On a more general note. I picked on cosine for this example, but I could have used any of the six trig functions, so be careful!

Powers of trig functions

Remember that if n is a positive integer then

$$\sin^n x = (\sin x)^n$$

The same holds for all the other trig functions as well of course. This is just a notational idiosyncrasy that you've got to get used to. Also remember to keep the following straight.

$$\tan^2 x \quad \text{vs.} \quad \tan x^2$$

In the first case we taking the tangent then squaring result and in the second we are squaring the x then taking the tangent.

The $\tan x^2$ is actually not the best notation for this type of problem, but I see people (both students and instructors) using it all the time. We really should probably use $\tan(x^2)$ to make things clear.

Inverse trig notation

The notation for inverse trig functions is not the best. You need to remember, that despite what I just got done talking about above,

$$\cos^{-1} x \neq \frac{1}{\cos x}$$

This is why I said that n was a positive integer in the previous discussion. I wanted to avoid this notational problem. The -1 in $\cos^{-1} x$ is NOT an exponent, it is there to denote the fact that we are dealing with an inverse trig function.

There is another notation for inverse trig functions that avoids this problem, but it is not always used.

$$\cos^{-1} x = \arccos x$$

Common Errors

This is a set of errors that really doesn't fit into any of the other topics so I included all them here.

Read the instructions!!!!!!

This is probably one of the biggest mistakes that students make. You've got to read the instructions and the problem statement carefully. Make sure you understand what you are being asked to do BEFORE you start working the problem

Far too often students run with the assumption : "It's in section X so they must want me to _____." In many cases you simply can't assume that. Do not just skim the instruction or read the first few words and assume you know the rest.

Instructions will often contain information pertaining to the steps that your instructor wants to see and the form the final answer must be in. Also, many math problems can proceed in several ways depending on one or two words in the problem statement. If you miss those one or two words, you may end up going down the wrong path and getting the problem completely wrong.

Not reading the instructions is probably the biggest source of point loss for my students.

Pay attention to restrictions on formulas

This is an error that is often compounded by instructors (me included on occasion, I must admit) that don't give or make a big deal about restrictions on formulas. In some cases the instructors forget the restrictions, in others they seem to have the idea that the restrictions are so obvious that they don't need to give them, and in other cases the instructors just don't want to be bothered with explaining the restrictions so they don't give them.

For instance, in an algebra class you should have run across the following formula.

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

The problem is there is a restriction on this formula and many instructors don't bother with it and so students aren't always aware of it. Even if instructors do give the restriction on this formula many students forget it as they are rarely faced with a case where the formula doesn't work.

Take a look at the following example to see what happens when the restriction is violated (I'll give the restriction at the end of example.)

1. $\sqrt{1} = \sqrt{1}$ This is certainly a true statement.
2. $\sqrt{(1)(1)} = \sqrt{(-1)(-1)}$ Since $1 = (1)(1)$ and $1 = (-1)(-1)$.
3. $\sqrt{1}\sqrt{1} = \sqrt{-1}\sqrt{-1}$ Use the above property on both roots.
4. $(1)(1) = (i)(i)$ Since $i = \sqrt{-1}$
5. $1 = i^2$ Just a little simplification.
6. $1 = -1$ Since $i^2 = -1$.

So clearly we've got a problem here as we are well aware that $1 \neq -1$! The problem arose in step 3. The property that I used has the restriction that a and b can't both be negative. It is okay if one or the other is negative, but they can't BOTH be negative!

Ignoring this kind of restriction can cause some real problems as the above example shows.

There is also an example from calculus of this kind of problem. If you haven't had calculus then you can skip this one. One of the more basic formulas that you'll get is

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

This is where most instructors leave it, despite the fact that there is a fairly important restriction that needs to be given as well. I suspect most instructors are so used to using the formula that they just implicitly feel that everyone knows the restriction and so don't have to give it. I know that I've done this myself here!

In order to use this formula n MUST be a fixed constant! In other words you can't use the formula to find the derivative of x^x since the exponent is not a fixed constant. If you tried to use the rule to find the derivative of x^x you would arrive at

$$x \cdot x^{x-1} = x^x$$

and the correct derivative is,

$$\frac{d}{dx}(x^x) = x^x(1 + \ln x)$$

So, you can see that what we got be incorrectly using the formula is not even close to the correct answer.

Changing your answer to match the known answer

Since I started writing my own homework problems I don't run into this as often as I used to, but it annoyed me so much that I thought I'd go ahead and include it.

In the past, I'd occasionally assign problems from the text with answers given in the back. Early in the semester I would get homework sets that had incorrect work but the correct answer just blindly copied out of the back. Rather than go back and find their mistake the students would just copy the correct answer down in the hope that I'd miss it while grading. While on occasion I'm sure that I did miss it, when I did catch it, it cost the students far more points than the original mistake would have cost them.

So, if you do happen to know what the answer is ahead of time and your answer doesn't match it **GO BACK AND FIND YOUR MISTAKE!!!!** Do not just write the correct answer down and hope. If you can't find your mistake then write down the answer you get, not the known and (hopefully) correct answer.

I can't speak for other instructors, but if I see the correct answer that isn't supported by your work you will lose far more points than the original mistake would have cost you had you just written down the incorrect answer.

Don't assume you'll do the work correctly and just write the answer down

This error is similar to the previous one in that it assumes that you have the known answer ahead of time.

Occasionally there are problems for which you can get the answer to intermediate step by looking at the known answer. In these cases do not just assume that your initial work is correct and write down the intermediate answer from the known answer without actually doing the work to get the answers to those intermediate steps.

Do the work and check your answers against the known answer to make sure you didn't make a mistake. If your work doesn't match the known answer then you know you made a mistake. Go back and find it.

There are certain problems in a differential equations class in which if you know the answer ahead of time you can get the roots of a quadratic equation that you must solve as well as the solution to a system of equations that you must also solve. I won't bore you with the details of these types of problems, but I once had a student who was notorious for this kind of error.

There was one problem in particular in which he had written down the quadratic equation and had made a very simple sign mistake, but he assumed that he would be able to solve

the quadratic equation without any problems so just wrote down the roots of the equation that he got by looking at the known answer. He then proceeded with the problem, made a couple more very simple and easy to catch mistakes and arrived at the system of equations that he needed to solve. Again, because of his mistakes it was the incorrect system, but he simply assumed he would solve it correctly if he had done the work and wrote down the answer he got by looking at the solution.

This student received almost no points on this problem because he decided that in a differential equations class solving a quadratic equation or a simple system of equations was beneath him and that he would do it correctly every time if he were to do the work. Therefore, he would skip the work and write down what he knew the answers to these intermediate steps to be by looking at the known answer. If he had simply done the work he would have realized he made a mistake and could have found the mistakes as they were typically easy to catch mistakes.

So, the moral of the story is **DO THE WORK**. Don't just assume that if you were to do the work you would get the correct answer. Do the work and if it's the same as the known answer then you did everything correctly, if not you made a mistake so go back and find it.

Does your answer make sense?

When you're done working problems go back and make sure that your answer makes sense. Often the problems are such that certain answers just won't make sense, so once you've gotten an answer ask yourself if it makes sense. If it doesn't make sense then you've probably made a mistake so go back and try to find it.

Here are a couple of examples that I've actually gotten from students over the years.

In an algebra class we would occasionally work interest problems where we would invest a certain amount of money in an account that earned interest at a specific rate for a specific number of year/months/days depending on the problem. First, if you are earning interest then the amount of money should grow, so if you end up with less than you started you've made a mistake. Likewise, if you only invest \$2000 for a couple of years at a small interest rate you shouldn't have a couple of billion dollars in the account after two years!

Back in my graduate student days I was teaching a trig class and we were going to try and determine the height of a very well known building on campus given the length of the shadow and the angle of the sun in the sky. I doubt that anyone in the class knew the actual height of the building, but they had to know that it wasn't over two miles tall! I actually got an answer that was over two miles. It clearly wasn't a correct answer, but instead of going back to find the mistake (a very simple mistake was made) the student circled the obviously incorrect answer and moved on to the next problem.

Often the mistake that gives an obviously incorrect answer is an easy one to find. So, check your answer and make sure that they make sense!

Check your work

I can not stress how important this one is! **CHECK YOUR WORK!** You will often catch simple mistakes by going back over your work. The best way to do this, although it's time consuming, is to put your work away then come back and rework all the problems and check your new answers to those previously gotten. This is time consuming and so can't always be done, but it is the best way to check your work.

If you don't have that kind of time available to you, then at least read through your work. You won't catch all the mistakes this way, but you might catch some of the more glaring mistakes.

Depending on your instructors beliefs about working groups you might want to check your answer against other students. Some instructors frown on this and want you to do all your work individually, but if your instructor doesn't mind this, it's a nice way to catch mistakes.

Guilt by association

The title here doesn't do a good job of describing the kinds of errors here, but once you see the kind of errors that I'm talking about you will understand it.

Too often students make the following logic errors. Since the following formula is true

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \quad \text{where } a \text{ and } b \text{ can't both be negative}$$

there must be a similar formula for $\sqrt{a+b}$. In other words, if the formula works for one algebraic operation (i.e. addition, subtraction, division, and/or multiplication) it must work for all. The problem is that this usually isn't true! In this case

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

Likewise, from calculus students make the mistake that because

$$(f + g)' = f' + g'$$

the same must be true for a product of functions. Again, however, it doesn't work that way!

$$(f g)' \neq (f')(g')$$

So, don't try to extend formulas that work for certain algebraic operations to all algebraic operations. If you were given a formula for certain algebraic operation, but not others there was a reason for that. In all likelihood it only works for those operations in which you were given the formula!

Rounding Errors

For some reason students seem to develop the attitude that everything must be rounded as much as possible. This has gone so far that I've actually had students who refused to work with decimals! Every answer was rounded to the nearest integer, regardless of how wrong that made the answer.

There are simply some problems where rounding too much can get you in trouble and seriously change the answer. The best example of this is interest problems. Here's a quick example.

Recall (provided you've seen this formula) that if you invest P dollars at an interest rate of r that is compounded m times per year, then after t years you will have A dollars where,

So, let's assume that we invest \$10,000 at an interest rate of 6.5% compounded monthly for 15 years. So, here's what we've got

$$P = 10,000$$

$$r = \frac{6.5}{100} = 0.065$$

$$m = 12$$

$$t = 15$$

Remember that interest the interest rate is always divided by 100! So, here's what we will have after 15 years.

$$\begin{aligned} A &= 10000 \left(1 + \frac{0.065}{12} \right)^{(12)(15)} \\ &= 10,000(1.005416667)^{180} \\ &= 10,000(2.644200977) \\ &= 26,442.00977 \\ &= 26,442.01 \end{aligned}$$

So, after 15 years we will have \$26,442.01. You will notice that I didn't round until the very last step and that was only because we were working with money which usually only has two decimal places. That is required in these problems. Here are some

examples of rounding to show you how much difference rounding too much can make. At each step I'll round each answer to the give number of decimal places.

First, I'll do the extreme case of no decimal places at all, i.e. only integers. This is an extreme case, but I've run across it occasionally.

$$\begin{aligned}
 A &= 10000 \left(1 + \frac{0.065}{12} \right)^{(12)(15)} \\
 &= 10,000(1)^{180} && 1.005416667=1 \text{ when rounded.} \\
 &= 10,000(1) \\
 &= 10,000.00
 \end{aligned}$$

It's extreme but it makes the point.

Now, I'll round to three decimal places.

$$\begin{aligned}
 A &= 10000 \left(1 + \frac{0.065}{12} \right)^{(12)(15)} \\
 &= 10,000(1.005)^{180} && 1.005416667=1.005 \text{ when rounded.} \\
 &= 10,000(2.454) && 2.454093562 = 2.454 \text{ when rounded.} \\
 &= 24,540.00
 \end{aligned}$$

Now, round to five decimal places.

$$\begin{aligned}
 A &= 10000 \left(1 + \frac{0.065}{12} \right)^{(12)(15)} \\
 &= 10,000(1.00542)^{180} && 1.005416667=1.00542 \text{ when rounded.} \\
 &= 10,000(2.64578) && 2.645779261=2.64578 \text{ when rounded.} \\
 &= 26,457.80
 \end{aligned}$$

Finally, round to seven decimal places.

$$\begin{aligned}
 A &= 10000 \left(1 + \frac{0.065}{12} \right)^{(12)(15)} \\
 &= 10,000(1.0054167)^{180} && 1.005416667=1.0054167 \text{ when rounded.} \\
 &= 10,000(2.6442166) && 2.644216599=2.6442166 \text{ when rounded.} \\
 &= 26,442.17
 \end{aligned}$$

I skipped a couple of possibilities in the computations. Here is a table of all possibilities from 0 decimal places to 8.

Decimal places of rounding	Amount after 15 years	Error in Answer
0	\$10,000.00	\$16,442.01 (Under)
1	\$10,000.00	\$16,442.01 (Under)
2	\$60,000.00	\$33,557.99 (Over)
3	\$24,540.00	\$1,902.01 (Under)
4	\$26,363.00	\$79.01 (Under)
5	\$26,457.80	\$15.79 (Over)
6	\$26,443.59	\$1.58 (Over)
7	\$26,442.17	\$0.16 (Over)
8	\$26,442.02	\$0.01 (Over)

So, notice that it takes at least 4 digits of rounding to start getting “close” to the actual answer. Note as well that in the world of business the answers we got with 4, 5, 6 and 7 decimal places of rounding would probably also be unacceptable. In a few cases (such as banks) where every penny counts even the last answer would also be unacceptable!

So, the point here is that you must be careful with rounding. There are some situations where too much rounding can drastically change the answer!

Bad notation

These are not really errors, but bad notation that always sets me on edge when I see it. Some instructors, including me after a while, will take off points for these things. This is just notational stuff that you should get out of the habit of writing if you do it. You should reach a certain mathematical “maturity” after awhile and not use this kind of notation.

First, I see the following all too often,

$$2 + x - 6x = 2 + -5x$$

The $+ - 5$ just makes no sense! It combines into a negative SO WRITE IT LIKE THAT! Here’s the correct way,

$$2 + x - 6x = 2 - 5x$$

This is the correct way to do it! I expect my students to do this as well.

Next, one (the number) times something is just the something, there is no reason to continue to write the one. For instance,

$$2 + 7x - 6x = 2 + x$$

Do not write this as $2 + 1x$! The coefficient of one is not needed here since $1x = x$! Do not write the coefficient of 1!

This same thing holds for an exponent of one anything to the first power is the anything so there is usually no reason to write the one down!

$$x^1 = x$$

In my classes, I will attempt to stop this behavior with comments initially, but if that isn't enough to stop it, I will start taking points off.

Calculus Errors

Many of the errors listed here are not really calculus errors, but errors that commonly occur in a calculus class and notational errors that are calculus related. If you haven't had a calculus class then I would suggest that you not bother with this section as it probably won't make a lot of sense to you.

If you are just starting a calculus class then I would also suggest that you be very careful with reading this. At some level this part is intended to be read by a student taking a calculus course as he/she is taking the course. In other words, after you've covered limits come back and look at the issues involving limits, then do the same after you've covered derivatives and then with integrals. Do not read this prior to the class and try to figure out how calculus works based on the few examples that I've given here! This will only cause you a great amount of grief down the road.

Derivatives and Integrals of Products/Quotients

Recall that while

$$(f \pm g)'(x) = f'(x) \pm g'(x) \qquad \int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$$

are true, the same thing can't be done for products and quotients. In other words,

$$\begin{aligned} (fg)'(x) &\neq f'(x)g'(x) & \int f(x)g(x)dx &\neq \left(\int f(x)dx\right)\left(\int g(x)dx\right) \\ \left(\frac{f}{g}\right)'(x) &\neq \frac{f'(x)}{g'(x)} & \int \frac{f(x)}{g(x)}dx &\neq \frac{\int f(x)dx}{\int g(x)dx} \end{aligned}$$

If you need convincing of this consider the example of $f(x) = x^4$ and $g(x) = x^{10}$.

$$\begin{aligned} (fg)'(x) &\neq f'(x)g'(x) \\ (x^4x^{10})' &\neq (x^4)'(x^{10})' \\ (x^{14})' &\neq (4x^3)(10x^9) \\ 14x^{13} &\neq 40x^{12} \end{aligned}$$

I only did the case of the derivative of a product, but clearly the two aren't equal! I'll leave it to you to check the remaining three cases if you'd like to.

Remember that in the case of derivatives we've got the product and quotient rule. In the case of integrals there are no such rules and when faced with an integral of a product or quotient they will have to be dealt with on a case by case basis.

Proper use of the formula for $\int x^n dx$

Many students forget that there is a restriction on this integration formula, so for the record here is the formula along with the restriction.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad \text{provided } n \neq -1$$

That restriction is incredibly important because if we allowed $n = -1$ we would get division by zero in the formula! Here is what I see far too many students do when faced with this integral.

$$\int x^{-1} dx = \frac{x^0}{0} + c = x^0 + c = 1 + c$$

THIS ISN'T TRUE!!!!!! There are all sorts of problems with this. First there's the improper use of the formula, then there is the division by zero problem! This should NEVER be done this way.

Recall that the correct integral of x^{-1} is,

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln |x| + c$$

This leads us to the next error.

Dropping the absolute value when integrating $\int \frac{1}{x} dx$

Recall that in the formula

$$\int \frac{1}{x} dx = \ln |x| + c$$

the absolute value bars on the argument are required! It is certainly true that on occasion they can be dropped after the integration is done, but they are required in most cases. For instance contrast the two integrals,

$$\int \frac{2x}{x^2+10} dx = \ln|x^2+10| + C = \ln(x^2+10) + c$$

$$\int \frac{2x}{x^2-10} dx = \ln|x^2-10| + c$$

In the first case the x^2 is positive and adding 10 on will not change that fact so since $x^2+10 > 0$ we can drop the absolute value bars. In the second case however, since we don't know what the value of x is, there is no way to know the sign of x^2-10 and so the absolute value bars are required.

Improper use of the formula $\int \frac{1}{x} dx = \ln|x| + c$

Gotten the impression yet that there are more than a few mistakes made by students when integrating $\frac{1}{x}$? I hope so, because many students lose huge amounts of points on these mistakes. This is the last one that I'll be covering however.

In this case, students seem to make the mistake of assuming that if $\frac{1}{x}$ integrates to $\ln|x|$ then so must one over anything! The following table gives some examples of incorrect uses of this formula.

Integral	Incorrect Answer	Correct Answer
$\int \frac{1}{x^2+1} dx$	$\ln(x^2+1) + c$	$\tan^{-1}(x) + c$
$\int \frac{1}{x^2} dx$	$\ln(x^2) + c$	$-x^{-1} + c = -\frac{1}{x} + c$
$\int \frac{1}{\cos x} dx$	$\ln \cos x + c$	$\ln \sec x + \tan x + c$

So, be careful when attempting to use this formula. This formula can only be used when the integral is of the form $\int \frac{1}{x} dx$. Often, an integral can be written in this form with an appropriate u -substitution (the two integrals from previous example for instance), but if it can't be then the integral will NOT use this formula so don't try to.

Improper use of Integration formulas in general

This one is really the same issue as the previous one, but so many students have trouble with logarithms that I wanted to treat that example separately to make the point.

So, as with the previous issue students tend to try and use “simple” formulas that they know to be true on integrals that, on the surface, kind of look the same. So, for instance we’ve got the following two formulas,

$$\int \sqrt{u} \, du = \frac{2}{3} u^{\frac{3}{2}} + C$$

$$\int u^2 \, du = \frac{1}{3} u^3 + C$$

The mistake here is to assume that if these are true then the following must also be true.

$$\int \sqrt{\text{anything}} \, du = \frac{2}{3} (\text{anything})^{\frac{3}{2}} + C$$

$$\int (\text{anything})^2 \, du = \frac{1}{3} (\text{anything})^3 + C$$

This just isn’t true! The first set of formulas work because it is the square root of a single variable or a single variable squared. If there is anything other than a single u under the square root or being squared then those formulas are worthless. On occasion these will hold for things other than a single u , but in general they won’t hold so be careful!

Here’s another table with a couple of examples of these formulas not being used correctly.

Integral	Incorrect Answer	Correct Answer
$\int \sqrt{x^2 + 1} \, dx$	$\frac{2}{3} (x^2 + 1)^{\frac{3}{2}} + C$	$\frac{1}{2} (x\sqrt{x^2 + 1} + \ln x + \sqrt{x^2 + 1}) + C$
$\int \cos^2 x \, dx$	$\frac{1}{3} \cos^3 x + C$	$\frac{x}{2} + \frac{1}{4} \sin(2x) + C$

If you aren’t convinced that the incorrect answers really aren’t correct then remember that you can always check you answers to indefinite integrals by differentiating the answer. If you did everything correctly you should get the function you originally integrated, although in each case it will take some simplification to get the answers to be the same.

Also, if you don’t see how to get the correct answer for these they typically show up in a Calculus II class. The second however, you could do with only Calculus I under your belt if you can remember an appropriate trig formula.

Dropping limit notation

The remainder of the errors in this document consists mostly of notational errors that students tend to make.

I'll start with limits. Students tend to get lazy and start dropping limit notation after the first step. For example, an incorrectly worked problem is

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3 = 6$$

There are several things wrong with this. First, when you drop the limit symbol you are saying that you've in fact taken the limit. So, in the first equality,

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3}$$

you are saying that the value of the limit is

$$\frac{(x - 3)(x + 3)}{x - 3}$$

and this is clearly not the case. Also, in the final equality,

$$x + 3 = 6$$

you are making the claim that each side is the same, but this is only true provided $x = 3$ and what you really are trying to say is

$$\lim_{x \rightarrow 3} x + 3 = 6$$

You may know what you mean, but someone else will have a very hard time deciphering your work. Also, your instructor will not know what you mean by this and won't know if you understand that the limit symbols are required in every step until you actually take the limit. If you are one of my students, I won't even try to read your mind and I will assume that you didn't understand and take points off accordingly.

So, while you may feel that it is silly and unnecessary to write limits down at every step it is proper notation and in my class I expect you to use proper notation. The correct way to work this limit is.

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3} x + 3 = 6$$

The limit is required at every step until you actually take the limit, at which point the limit must be dropped as I have done above.

Improper derivative notation

When asked to differentiate $f(x) = x(x^3 - 2)$ I will get the following for an answer on occasion.

$$f(x) = x(x^3 - 2) = x^4 - 2x = 4x^3 - 2$$

This is again a situation where you may know what you're intending to say here, but anyone else who reads this will come away with the idea that $x^4 - 2x = 4x^3 - 2$ and that is clearly NOT what you are trying to say. However, it IS what you are saying when you write it this way.

The proper notation is

$$f(x) = x(x^3 - 2) = x^4 - 2x$$

$$f'(x) = 4x^3 - 2$$

Loss of integration notation

There are many dropped notation errors that occur with integrals. Let's start with this example.

$$\int x(3x - 2)dx = 3x^2 - 2x = x^3 - x^2 + c$$

As with the derivative example above, both of these equalities are incorrect. The minute you drop the integral sign you are saying that you've done the integral! So, this means that the first equality is saying that the value of the integral is $3x^2 - 2x$, when in reality all you're doing is simplifying the function. Likewise, the last equality says that the two functions, $3x^2 - 2x$ and $x^3 - x^2 + c$ are equal, when they are not! Here is the correct way to work this problem.

$$\int x(3x - 2)dx = \int 3x^2 - 2x dx = x^3 - x^2 + c$$

Another big problem in dropped notation is students dropping the dx at the end of the integrals. For instance,

$$\int 3x^2 - 2x$$

The problem with this is that the dx tells us where the integral stops! So, this can mean a couple of different things.

$$\int 3x^2 - 2x dx = x^3 - x^2 + c \quad \text{OR}$$

$$\int 3x^2 dx - 2x = x^3 - 2x + c$$

Without the dx a reader is left to try and intuit where exactly the integral ends! The best way to think of this is that parenthesis always come in pairs "(" and ")". You don't open a set of parenthesis without closing it. Likewise, \int is always paired up with a dx . You can always think of \int as the opening parenthesis and the dx as the closing parenthesis.

Another dropped notation error that I see on a regular basis is with definite integrals. Students tend to drop the limits of integration after the first step and do the rest of the problem with *implied* limits of integration as follows.

$$\int_1^2 x(3x - 2)dx = \int 3x^2 - 2x dx = x^3 - x^2 = 8 - 4 - (1 - 1) = 4$$

Again, the first equality here just doesn't make sense! The answer to a definite integral is a number, while the answer to an indefinite integral is a function. When written as above you are saying the answer to the definite integral and the answer to the indefinite integral are the same when they clearly aren't!

Likewise, the second to last equality just doesn't make sense. Here you are saying that the function, $x^3 - x^2$ is equal to $8 - 4 - (1 - 1) = 4$ and again, this just isn't true! Here is the correct way to work this problem.

$$\int_1^2 x(3x-2)dx = \int_1^2 3x^2 - 2x dx = (x^3 - x^2) \Big|_1^2 = 8 - 4 - (1 - 1) = 4$$

Loss of notation in general

The previous three topics that I've discussed have all been examples of dropped notation errors that students first learning calculus tend to make on a regular basis. Be careful with these kinds of errors. You may know what you're trying to say, but improper notation may imply something totally different.

Remember that in many ways written mathematics is like a language. If you mean to say to someone

“I'm thirsty, could you please get me a glass of water to drink.”

You wouldn't drop words that you considered extraneous to the message and just say

“Thirsty, drink”

This is meaningless and the person that you were talking to may get the idea that you are thirsty and wanted to drink something. They would definitely not get the idea that you wanted water to drink or that you were asking them to get it for you. You would know that is what you wanted, but those two words would not convey that to anyone else.

This may seem like a silly example to you, because you would never do something like this. You would give the whole sentence and not just two words because you are fully aware of how confusing simply saying those two words would be. That, however, is exactly the point of the example.

You know better than to skip important words in spoken language, so you shouldn't skip important notation (i.e. words) in writing down the language of mathematics. You may feel that they aren't important parts to the message, but they are. Anyone else reading the message you wrote down would not necessarily know that you neglected to write down those important pieces of notation and would very likely misread the message you were trying to impart.

So, be careful with proper notation. In my class, I grade the “message” you write down not the “message” that you meant to impart. I can’t read your mind so I don’t even try to. If the “message” that I read in grading your homework or exam is wrong, I will grade it appropriately.

Dropped constant of integration

Dropping the constant of integration on indefinite integrals (the $+ c$ part) is one of the biggest errors that students make in integration. There are actually two errors here that students make. Some students just don’t put it in at all, and others drop it from intermediate steps and then just tack it onto the final answer.

Those that don’t include it at all tend to be the students that don’t remember (or never really understood) that the indefinite integrals give the most general possible function that we could differentiate to get the integrand (the function we integrated). Because it is the most general possible function we’ve got to include the constant, since constants differentiate to zero.

For those that drop it from all intermediate steps and just tack it on at the end there are other issues. I suppose that the problem is these (in fact it’s probably most) students just don’t see why it’s important to include the constant of integration. This is partially a problem with the class itself. Calculus classes just don’t really have good examples of why the constant of integration is so important or how it comes into play in later steps.

The first place where constants of integration play a major role is a first course in differential equations. Here the constant of integration will show up in the middle of the problem. If it’s dropped there and then just added back in on the final answer or not put in at all, the answer will be very wrong. The answer won’t be wrong because the instructor said that it was wrong without the constant or because it was only added in at the last step. The answer will be wrong because the function you get without dropping it will be totally different from the function you get if you do drop it!

Misconceptions about $\frac{1}{0}$ and $\frac{1}{\infty}$

This is not so much about an actual error that students make, but instead a misconception that can, on occasion, lead to errors. This is also a misconception that is often encouraged by laziness on the part of the instructor.

So, just what is this misconception? Often, we will write $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$. The problem is that neither of these are technically correct and in fact the second, depending on the situation, can actually be $\frac{1}{0} = -\infty$. All three of these are really limits and we just short hand them. What we really should write is

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

In the first case 1 over something increasingly large is increasingly small and so **in the limit** we get zero. In the last two cases note that we've got to use one-sided limits as $\lim_{x \rightarrow 0} \frac{1}{x}$ doesn't even exist! In these two cases, 1 over something increasingly small is increasingly large and will have the sign of the denominator and so in the limit it goes to either ∞ or $-\infty$.

Indeterminate forms

This is actually a generalization of the previous topic. The two operations above, $\infty - \infty$ and $\frac{\infty}{\infty}$ are called *indeterminate forms* because there is no one single value for them.

Depending on the situation they have a very wide range of possible answers.

There are many more indeterminate forms that you need to look out for. As with the previous discussion there is no way to determine their value without taking the situation into consideration. Here are a few of the more common indeterminate forms.

$$\begin{array}{cccc} \infty - \infty & \frac{\infty}{\infty} & \frac{0}{0} & 0 \cdot \infty \\ 0^0 & 1^\infty & \infty^0 & \end{array}$$

Let's just take a brief look at 0^0 to see the potential problems. Here we really have two separate rules that are at odds with each other. Typically we have $0^n = 0$ (provided n is positive) and $a^0 = 1$. Each of these rules implies that we could get different answers. Depending on the situation we could get either 0 or 1 as an answer here. In fact, it's also possible to get something totally different from 0 or 1 as an answer here as well.

All the others listed here have similar problems. So, when dealing with indeterminate forms you need to be careful and not jump to conclusions about the value.

Treating infinity as a number

In the following discussion I'm going to be working exclusively with real numbers (things can be different with say complex numbers). I'm also going to think of infinity (∞) as a really, really large number. This is not technically accurate as infinity is really a concept to denote a state of endlessness or a state of no limits in any direction. In terms

of a number line infinity (∞) denotes moving in the positive direction without ever stopping. Likewise, negative infinity ($-\infty$) on a number line denotes moving in the negative direction without ever stopping.

The problem with the conceptual definition of infinity is that many students have a hard time dealing with arithmetic involving infinity when they think of it in terms of its conceptual definition. However, if we simply call it a really, really large number it seems to help a little so that's how I'm going to think of it for the purposes of this discussion.

Most students have run across infinity at some point in time prior to a calculus class. However, when they have dealt with it, it was just a symbol used to represent a really, really large positive or negative number and that was the extent of it. Once they get into a calculus class students are asked to do some basic algebra with infinity and this is where they get into trouble. Infinity is NOT a number and for the most part doesn't behave like a number. When you add two non-zero numbers you get a new number. For example, $4 + 7 = 11$. With infinity this is not true. With infinity you have the following.

$$\infty + a = \infty \quad \text{where } a \neq -\infty$$

$$\infty + \infty = \infty$$

In other words, a really, really large positive number (∞) plus any positive number, regardless of the size, is still a really, really large positive number. Likewise, you can add a negative number (*i.e.* $a < 0$) to a really, really large positive number and stay really, really large and positive. So, addition involving infinity can be dealt with in an intuitive way if you're careful. Note as well that the a must NOT be negative infinity. If it is, there are some serious issues that we need to deal with.

Subtraction with negative infinity can also be dealt with in an intuitive way. A really, really large negative number minus any positive number, regardless of its size, is still a really, really large negative number. Subtracting a negative number (*i.e.* $a < 0$) from a really, really large negative number will still be a really, really large negative number. Or,

$$-\infty - a = -\infty \quad \text{where } a \neq -\infty$$

$$-\infty - \infty = -\infty$$

Again, a must not be negative infinity to avoid some potentially serious difficulties.

Multiplication can also be dealt with fairly intuitively. A really, really large number (positive, or negative) times any number, regardless of size, is still a really, really large number. In the case of multiplication we have

$$(a)(\infty) = \infty \quad \text{if } a > 0$$

$$(a)(\infty) = -\infty \quad \text{if } a < 0$$

$$(\infty)(\infty) = \infty$$

$$(-\infty)(-\infty) = \infty$$

$$(-\infty)(\infty) = -\infty$$

What you know about products of positive and negative numbers is still true.

Some forms of division can be dealt with intuitively as well. A really, really large number divided by a number that isn't too large is still a really, really large number.

$$\frac{\infty}{a} = \infty \quad \text{if } a > 0$$

$$\frac{\infty}{a} = -\infty \quad \text{if } a < 0$$

$$\frac{-\infty}{a} = -\infty \quad \text{if } a > 0$$

$$\frac{-\infty}{a} = \infty \quad \text{if } a < 0$$

Division of a number by infinity is somewhat intuitive, but there are a couple of subtleties that you need to be aware of. I go into this in more detail in the section about

Misconceptions about $\frac{1}{0}$ and $\frac{1}{\infty}$ above, but one way to think of it is the following. A number that isn't too large divided by infinity (a really, really large number) is a very, very, very small number. In other words,

$$\frac{a}{\infty} = 0$$

$$\frac{a}{-\infty} = 0$$

So, I've dealt with almost every basic algebraic operation involving infinity. There are two cases that that I haven't dealt with yet. These are

$$\infty - \infty = ?$$

$$\frac{\pm \infty}{\pm \infty} = ?$$

The problem with these two is that intuition doesn't really help here. A really, really large number minus a really, really large number can be anything ($-\infty$, a constant, or ∞). Likewise, a really, really large number divided by a really, really large number can also be anything ($\pm \infty$ - this depends on sign issues, 0, or a non-zero constant).

What you've got to remember here is that there are really, really large numbers and then there are really, really, really large numbers. In other words, some infinities are larger than other infinities. With addition, multiplication and the first sets of division I worked this isn't an issue. The general size of the infinity just doesn't affect the answer. However, with the subtraction and division I listed above, it does matter as you will see.

Here is one way to think of this idea that some infinities are larger than others. This is a fairly dry and technical way to think of this and your calculus problems will probably

never use this stuff, but this it is a nice way of looking at this. Also, please note that I'm not trying to give a precise proof of anything here. I'm just trying to give you a little insight into the problems with infinity and how some infinities can be thought of as larger than others. For a much better (and definitely more precise) discussion see,

<http://www.math.vanderbilt.edu/~schectex/courses/infinity.pdf>

Let's start by looking at how many integers there are. Clearly, I hope, there are an infinite number of them, but let's try to get a better grasp on the "size" of this infinity. So, pick any two integers completely at random. Start at the smaller of the two and list, in increasing order, all the integers that come after that. Eventually we will reach the larger of the two integers that you picked.

Depending on the relative size of the two integers it might take a very, very long time to list all the integers between them and there isn't really a purpose to doing it. But, it could be done if we wanted to and that's the important part.

Because we could list all these integers between two randomly chosen integers we say that the integers are *countably infinite*. Again, there is no real reason to actually do this, it is simply something that can be done if we should chose to do so.

In general a set of numbers is called countably infinite if we can find a way to list them all out. In a more precise mathematical setting this is generally done with a special kind of function called a *bijection* that associates each number in the set with exactly one of the positive integers. To see some more details of this see the pdf given above.

It can also be shown that the set of all fractions are also countably infinite, although this is a little harder to show and is not really the purpose of this discussion. To see a proof of this see the pdf given above. It has a very nice proof of this fact.

Let's contrast this by trying to figure out how many numbers there are in the interval (0,1). By numbers, I mean all possible fractions that lie between zero and one as well as all possible decimals (that aren't fractions) that lie between zero and one. The following is similar to the proof given in the pdf above, but was nice enough and easy enough (I hope) that I wanted to include it here.

To start let's assume that all the numbers in the interval (0,1) are countably infinite. This means that there should be a way to list all of them out. We could have something like the following,

$$\begin{aligned}x_1 &= 0.692096\dots \\x_2 &= 0.171034\dots \\x_3 &= 0.993671\dots \\x_4 &= 0.045908\dots \\&\vdots \quad \quad \quad \vdots\end{aligned}$$

Now, select the i^{th} decimal out of x_i as shown below

$$x_1 = 0.\underline{6}92096\dots$$

$$x_2 = 0.17\underline{1}034\dots$$

$$x_3 = 0.99\underline{3}671\dots$$

$$x_4 = 0.045\underline{9}08\dots$$

$$\vdots \quad \quad \quad \vdots$$

and form a new number with these digits. So, for our example we would have the number

$$x = 0.6739\dots$$

In this new decimal replace all the 3's with a 1 and then replace every other number with a 3. In the case of our example this would yield the new number

$$\bar{x} = 0.3313\dots$$

Notice that this number is in the interval $(0,1)$ and also notice that given how we choose the digits of the number this number will not be equal to the first number in our list, x_1 , because the first digit of each is guaranteed to not be the same. Likewise, this new number will not get the same number as the second in our list, x_2 , because the second digit of each is guaranteed to not be the same. Continuing in this manner we can see that this new number we constructed, \bar{x} , is guaranteed to not be in our listing. But this contradicts the initial assumption that we could list out all the numbers in the interval $(0,1)$. Hence, it must not be possible to list out all the numbers in the interval $(0,1)$.

Sets of numbers, such as all the numbers in $(0,1)$, that we can't write down in a list are called *uncountably infinite*.

The reason for going over this is the following. An infinity that is uncountably infinite is significantly larger than an infinity that is only countably infinite. So, if we take the difference of two infinities we have a couple of possibilities.

$$\infty(\text{uncountable}) - \infty(\text{countable}) = \infty$$

$$\infty(\text{countable}) - \infty(\text{uncountable}) = -\infty$$

Notice that we didn't put down a difference of two infinities of the same type. Depending upon the context there might still have some ambiguity about just what the answer would be in this case, but that is a whole different topic. We could also do something similar for quotients of infinities.

$$\frac{\infty(\text{countable})}{\infty(\text{uncountable})} = 0$$
$$\frac{\infty(\text{uncountable})}{\infty(\text{countable})} = \infty$$

Again, we avoided a quotient of two infinities of the same type since, again depending upon the context, there might still be ambiguities about its value.