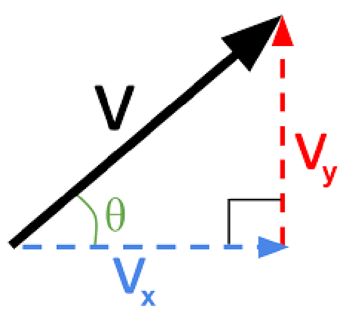
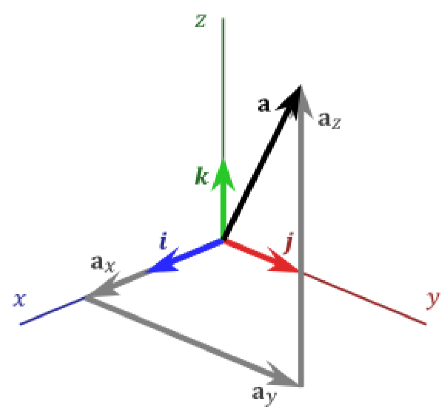
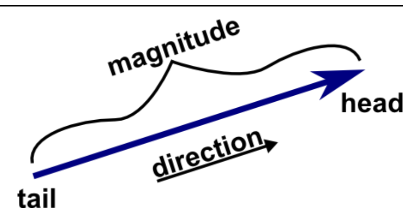
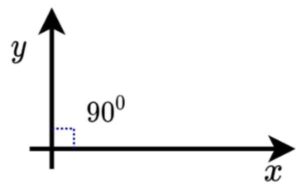
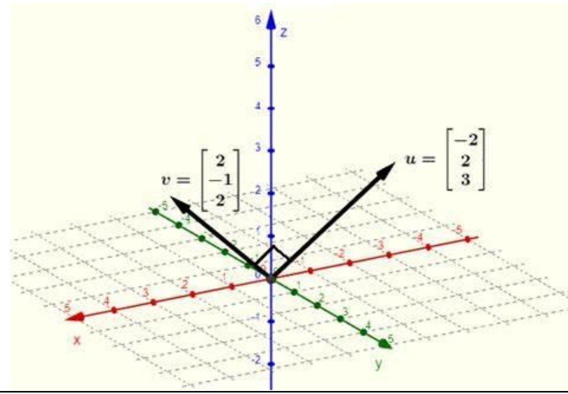
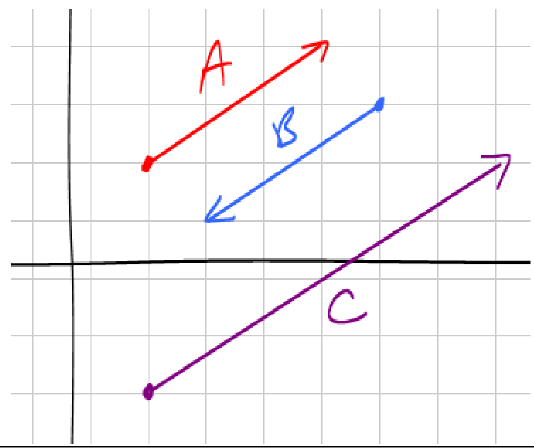
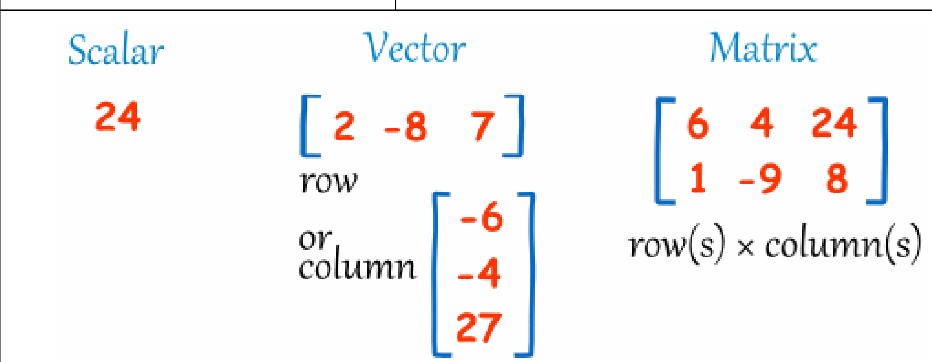
Harold's Vectors Cheat Sheet

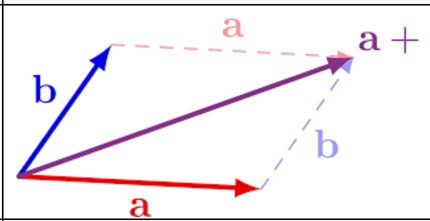
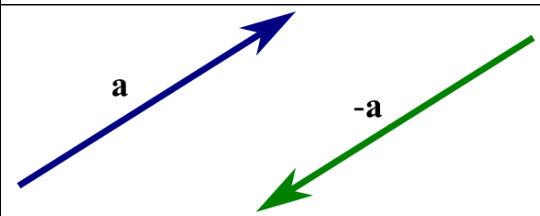
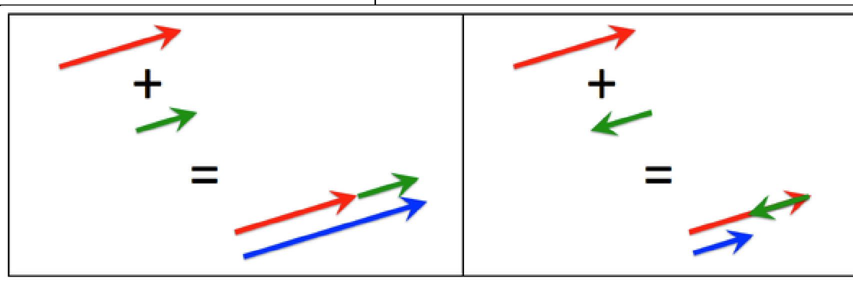
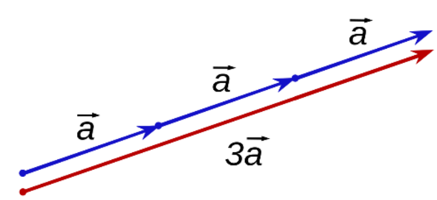
5 December 2022

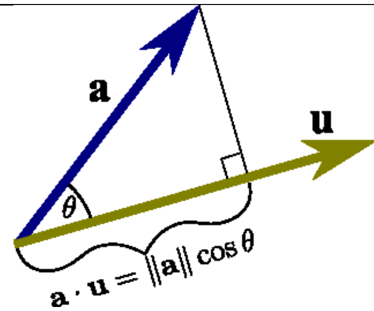
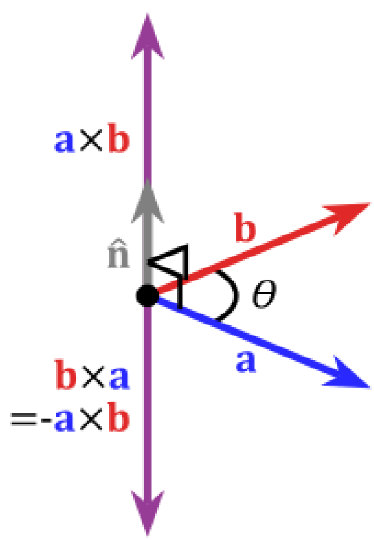
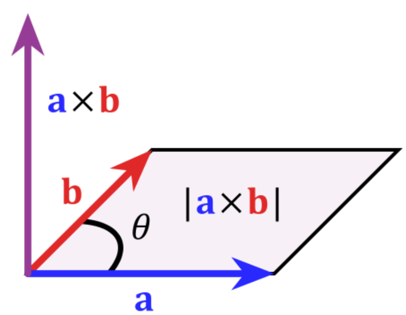
Definitions

Term	Formula	Example
Vector Notation	$\mathbf{A}, \boldsymbol{a}$	Bold letter
	$\vec{a}, \vec{\boldsymbol{a}}$	Arrow on top
Component Notation	$\boldsymbol{a} = \vec{a} = a_x\boldsymbol{\hat{\imath}} + a_y\boldsymbol{\hat{\jmath}} + a_z\boldsymbol{\hat{k}}$	$\boldsymbol{a} = 3\hat{\imath} + 4\hat{\jmath} + 5\boldsymbol{\hat{k}}$
	$\boldsymbol{a} = \langle a_x, a_y, a_z \rangle$	$\boldsymbol{a} = \langle 3, 4, 5 \rangle$
	$\boldsymbol{a} = r\angle\theta$	$\boldsymbol{a} = 5 \angle 53.13°$ (2D)
	 **2D**	 **3D**
Vectors Used in Examples	$\boldsymbol{a} = a_x\boldsymbol{\hat{\imath}} + a_y\boldsymbol{\hat{\jmath}} + a_z\boldsymbol{\hat{k}}$ $\boldsymbol{b} = b_x\boldsymbol{\hat{\imath}} + b_y\boldsymbol{\hat{\jmath}} + b_z\boldsymbol{\hat{k}}$ **2D**: set $a_z = b_z = 0$	$\boldsymbol{a} = 3\hat{\imath} + 4\hat{\jmath} + 5\boldsymbol{\hat{k}}$ $\boldsymbol{b} = 6\hat{\imath} - 7\hat{\jmath} - 8\boldsymbol{\hat{k}}$
Magnitude	$\lVert\boldsymbol{a}\rVert = \sqrt{a_x^2 + a_y^2 + a_z^2}$	$\lVert\boldsymbol{a}\rVert = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$
	Can also use $\lvert\boldsymbol{a}\rvert$. Length of vector, but with no direction (scalar). Similar to a hypotenuse. Think multi-dimensional Pythagorean Theorem.	
Direction	Divided into dimensional components.	A scalar with a direction is a vector. Example: speed vs. velocity
	$\tan\theta = \dfrac{a_y}{a_x}$	$\theta = \tan^{-1}\left(\dfrac{a_y}{a_x}\right) = \tan^{-1}\left(\dfrac{4}{3}\right) \cong 53.13°$
Unit Vector (Basis Vector)	$\boldsymbol{\hat{\imath}} = x - axis = \langle 1, 0, 0 \rangle$ $\boldsymbol{\hat{\jmath}} = y - axis = \langle 0, 1, 0 \rangle$ $\boldsymbol{\hat{k}} = z - axis = \langle 0, 0, 1 \rangle$	Circumflex or "hat" on top. Indicates direction only. Always has a magnitude of one (1 or unit).
	$\vec{\boldsymbol{u}} = \dfrac{\boldsymbol{a}}{\lVert\boldsymbol{a}\rVert}$	$\vec{\boldsymbol{u}} = \dfrac{a_x\boldsymbol{\hat{\imath}} + a_y\boldsymbol{\hat{\jmath}} + a_z\boldsymbol{\hat{k}}}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$

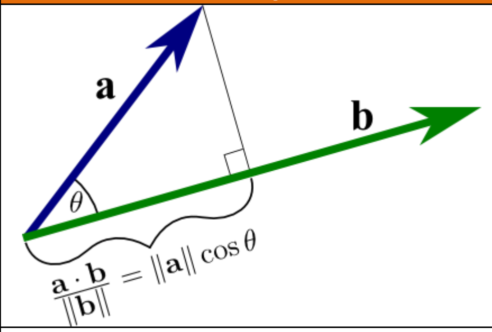
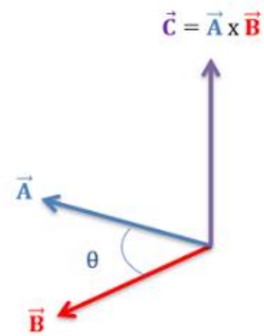
Scalar	*k, m*	A number with no direction or units.
Orthogonal	A change in one dimension does not change in any of the values in the other dimensions.	2D: right angle **Rectangular Coordinates:** The x-axis, y-axis, and z-axis are orthogonal to each other. **Polar Coordinates:** The angle is orthogonal to the line segment length
	if $\boldsymbol{a} \bullet \boldsymbol{b} = 0$	Two vectors are orthogonal if their dot product is zero.
		
Parallel	If $\boldsymbol{a} = k\boldsymbol{b}$	Two vectors are parallel if they have the same direction.
	Collinear of in opposite directions	
Vector vs. Matrix	vector = 1 x n or n x 1 matrix	A matrix with only one (1) row or column.
		

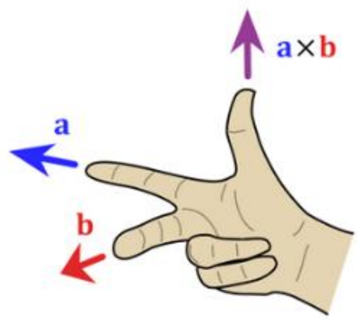
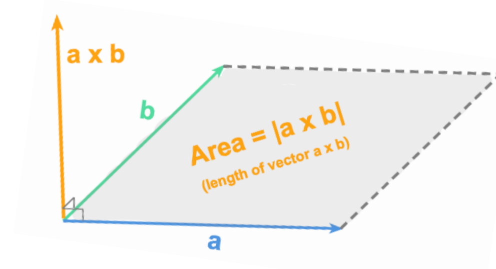
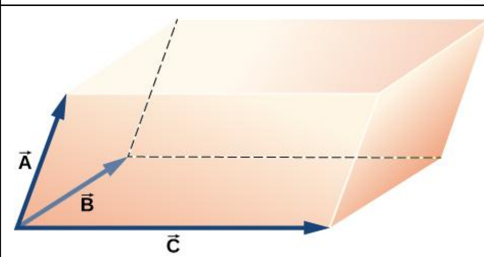
Vector Operations

Operation	Formula	Example
Addition	$\boldsymbol{a}+\boldsymbol{b}= (a_x+b_x)\hat{\boldsymbol{\imath}} + (a_y+b_y)\hat{\boldsymbol{\jmath}} + (a_z+b_z)\hat{\boldsymbol{k}}$	$\boldsymbol{a}+\boldsymbol{b}=(3+6)\hat{\boldsymbol{\imath}}+(4-7)\hat{\boldsymbol{\jmath}}+(5-8)\hat{\boldsymbol{k}} = 9\hat{\boldsymbol{\imath}}-3\hat{\boldsymbol{\jmath}}-3\hat{\boldsymbol{k}}$
	$\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$	Commutative
	$(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$	Associative
	$(k+m)\boldsymbol{a}=k\boldsymbol{a}+m\boldsymbol{a}$	Distributive
	$k(\boldsymbol{a}+\boldsymbol{b})=k\boldsymbol{a}+k\boldsymbol{b}$	
		
Subtraction	$\boldsymbol{a}-\boldsymbol{b}=\boldsymbol{a}+(-\boldsymbol{b})$	Change the direction of $\vec{b}$ then add.
		
Scalar Multiplication	$k\boldsymbol{a}=ka_x\hat{\boldsymbol{\imath}}+ka_y\hat{\boldsymbol{\jmath}}+ka_z\hat{\boldsymbol{k}}$	$3\cdot\boldsymbol{a}=3\cdot 3\hat{\boldsymbol{\imath}}+3\cdot 4\hat{\boldsymbol{\jmath}}+3\cdot 5\hat{\boldsymbol{k}} = 9\hat{\imath}+12\hat{\jmath}+15\hat{\boldsymbol{k}}$
	Changes the magnitude only.	
	$k(\boldsymbol{a}+\boldsymbol{b})=k\boldsymbol{a}+k\boldsymbol{b}$	
Dot Product (Scalar Product)	$\boldsymbol{a}\bullet\boldsymbol{b}=a_xb_x+a_yb_y+a_zb_z$	$\boldsymbol{a}\bullet\boldsymbol{b}=(3)(6)+(4)(-7)+(5)(-8) = -50$
	$\boldsymbol{a}\bullet\boldsymbol{b}=\lVert\boldsymbol{a}\rVert\,\lVert\boldsymbol{b}\rVert\cos\theta$	$\cos\theta=\dfrac{\boldsymbol{a}\bullet\boldsymbol{b}}{\lVert\boldsymbol{a}\rVert\,\lVert\boldsymbol{b}\rVert}$
	$\boldsymbol{a}\bullet\boldsymbol{b}=\boldsymbol{b}\bullet\boldsymbol{a}$	Commutative
	$\boldsymbol{a}\bullet(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\bullet\boldsymbol{b}+\boldsymbol{a}\bullet\boldsymbol{c}$	Distributive
	$k(\boldsymbol{a}\bullet\boldsymbol{b})=k\boldsymbol{a}\bullet\boldsymbol{b}=\boldsymbol{a}\bullet k\boldsymbol{b}$	Scalar Multiplication
	$0\bullet\boldsymbol{u}=0$	Zero Vector Dot Product
	$\boldsymbol{u}\bullet\boldsymbol{u}=\lVert\boldsymbol{u}\rVert^2$	Dot Product and Vector Magnitude Relationship
	It is always a scalar.	

		
Cross Product (Vector Product)	$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \boldsymbol{k}$ $= (a_y b_z - b_y a_z)\,\boldsymbol{i} - (a_x b_z - b_x a_z)\,\boldsymbol{j} + (a_x b_y - b_x a_y)\,\boldsymbol{k}$	
	$\lVert \boldsymbol{a} \times \boldsymbol{b} \rVert = \lVert \boldsymbol{a} \rVert \, \lVert \boldsymbol{b} \rVert \sin\theta$	$\sin\theta = \dfrac{\lVert \boldsymbol{a} \times \boldsymbol{b} \rVert}{\lVert \boldsymbol{a} \rVert \, \lVert \boldsymbol{b} \rVert}$
	$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$	Anti-Commutative
	$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} \neq \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \bullet \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \bullet \boldsymbol{b})\,\boldsymbol{c}$	Not Commutative
	$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ $\boldsymbol{a} \bullet (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \bullet \boldsymbol{c}$	Not Associative
	$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$ $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}$	Distributive
	$k(\boldsymbol{a} \times \boldsymbol{b}) = k\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times k\boldsymbol{b}$ $(k\boldsymbol{a}) \times \boldsymbol{b} = k(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \times (k\boldsymbol{b})$	Scalar Multiplication
	It is always a vector orthogonal to the other two vectors.	
		
Scalar Triple Product	$(\boldsymbol{a} \times \boldsymbol{b}) \bullet \boldsymbol{c} = \boldsymbol{a} \bullet (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \bullet (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \bullet (\boldsymbol{a} \times \boldsymbol{b}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$	
Vector Triple Product	$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \bullet \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a} \bullet \boldsymbol{b})\,\boldsymbol{c}$	

Vector Applications

<table>
<tr><th>Application</th><th>Formula</th><th>Example</th></tr>
<tr><td>Projection</td><td>$proj_{\boldsymbol{a}}\boldsymbol{b} = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{a}}\right)\boldsymbol{a}$</td><td>
</td></tr>
<tr><td>Right Hand Rule</td><td>The cross product produces a vector orthogonal to the other two vectors.</td><td>Use the right hand rule to determine direction of the cross product vector.</td></tr>
<tr><td></td><td colspan="2">


</td></tr>
<tr><td>Area
(Parallelagram)</td><td>$A = \|\boldsymbol{a} \times \boldsymbol{b}\|$</td><td>
</td></tr>
<tr><td>Volume
(Parallelepiped)</td><td>$V = \|(\boldsymbol{a} \times \boldsymbol{b}) \bullet \boldsymbol{c}\|$</td><td>
</td></tr>
<tr><td>Torque</td><td>$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$</td><td>$\|\boldsymbol{\tau}\| = r\,F\sin\theta$</td></tr>
<tr><td>Coplanar</td><td>Three vectors are coplanar if
$(\boldsymbol{a} \times \boldsymbol{b}) \bullet \boldsymbol{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = 0$</td><td>All three vectors are in the same plane.</td></tr>
</table>

