

## Harold's Trig Proofs

### Cheat Sheet

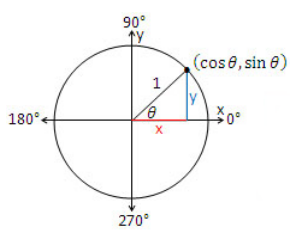
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After proving only three (3) trig formulas, we can easily derive **ALL** the trig formulas!

1. Pythagorean Identity
2. Sum and difference formula for sine
3. Sum and difference formula for cosine

### Proof of Pythagorean Identities

#### Proof

Given	
	Pythagorean Theorem $x^2 + y^2 = r^2$ $r = 1$ $\sin \theta = \frac{\textit{opposite}}{\textit{hypotenuse}} = \frac{y}{r} = \frac{y}{1} = y$ $\cos \theta = \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{x}{r} = \frac{x}{1} = x$
Substitute and Simplify	$\sin^2 \theta + \cos^2 \theta = 1^2$
Formula	<b><math>\sin^2 \theta + \cos^2 \theta = 1</math> [1]</b>

#### Proof

Given	$\sin^2 \theta + \cos^2 \theta = 1$ [1]
Divide by $\cos^2 \theta$ , then Simplify	$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$
Formula	<b><math>\tan^2 \theta + 1 = \sec^2 \theta</math> [2]</b>

#### Proof

Given	$\sin^2 \theta + \cos^2 \theta = 1$ [1]
Divide by $\sin^2 \theta$ , then Simplify	$\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$
Formula	<b><math>1 + \cot^2 \theta = \csc^2 \theta</math> [3]</b>

Proof of Sum and Difference Formulas	
Trig Sum and Difference Formulas	$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$ $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$
Proof Diagram	

Proof of $\sin(\alpha \pm \beta)$	
<b>Prove Sum</b>	
Given	$\sin(\alpha + \beta) = \frac{ED}{DA} = \frac{\text{opposite}}{\text{hypotenuse}}$
Alternate interior angles are congruent	$\alpha = \angle CAB = \angle HFA = \angle HDF$
Tallest vertical line	$ED = GF + HD$
Substitute, then divide and multiply by $AF$ & $FD$	$\sin(\alpha + \beta) = \frac{ED}{AD} = \frac{GF}{AD} + \frac{HD}{AD} = \frac{GF}{AF} \frac{AF}{AD} + \frac{HD}{FD} \frac{FD}{AD}$
Convert back to trig formulas	$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ [4]
<b>Prove Difference</b>	
Replace $+\beta$ with $-\beta$	$\cos(-\beta) = \cos(\beta)$ $\sin(-\beta) = -\sin(\beta)$
Simplify	$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$ [5]
General Formula [4+5]	$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$ [6]

<b>Proof of <math>\cos(\alpha \pm \beta)</math></b>	
<b>Prove Sum</b>	
Given	$\cos(\alpha + \beta) = \frac{AE}{AD} = \frac{\text{adjacent}}{\text{hypotenuse}}$
Longest horizontal line	$EA = GA - FH$
Substitute, then divide and multiply by <b>AF</b> & <b>DF</b>	$\cos(\alpha + \beta) = \frac{EA}{AD} = \frac{GA}{AD} - \frac{FH}{AD} = \frac{GA}{AF} \frac{AF}{AD} - \frac{FH}{DF} \frac{DF}{AD}$
Convert back to trig formulas	$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ [7]
<b>Prove Difference</b>	
Replace $+\beta$ with $-\beta$	$\cos(-\beta) = \cos(\beta)$ $\sin(-\beta) = -\sin(\beta)$
Simplify	$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ [8]
General Formula [7+8]	<b><math>\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta</math></b> [9]

<b>Proof of <math>\tan(\alpha \pm \beta)</math></b>	
<b>Prove Sum and Difference</b>	
Given	$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)}$
Substitute	$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$ [6] $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$ [9]
Divide by $(\cos \alpha \cos \beta)$ , then Simplify	$\tan(\alpha \pm \beta) = \frac{\sin \alpha \cos \beta \pm \cos \alpha \sin \beta}{\cos \alpha \cos \beta \mp \sin \alpha \sin \beta}$
General Formula	<b><math>\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}</math></b> [10]

## Proof of Double Angle Formulas (2θ)

### Proof

Given	$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ [4]
Substitute	$\theta = \alpha = \beta$
Simplify	$\sin(\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta$
Formula	<b><math>\sin(2\theta) = 2 \sin \theta \cos \theta</math></b> [14]

### Proof

Given	$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ [7]
Substitute	$\theta = \alpha = \beta$
Simplify	$\cos(\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta$
Formula	<b><math>\cos(2\theta) = \cos^2 \theta - \sin^2 \theta</math></b> [15]

### Proof

Given	$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$ [15] $\sin^2 \theta + \cos^2 \theta = 1$ [1]
Substitute	$\sin^2 \theta = 1 - \cos^2 \theta$
Simplify	$\cos(2\theta) = \cos^2 \theta - (1 - \cos^2 \theta)$
Formula	<b><math>\cos(2\theta) = 2 \cos^2 \theta - 1</math></b> [16]

### Proof

Given	$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$ [15] $\sin^2 \theta + \cos^2 \theta = 1$ [1]
Substitute	$\cos^2 \theta = 1 - \sin^2 \theta$
Simplify	$\cos(2\theta) = (1 - \sin^2 \theta) - \sin^2 \theta$
Formula	<b><math>\cos(2\theta) = 1 - 2 \sin^2 \theta</math></b> [17]

### Proof

Given	$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)}$
Substitute	$\sin(2\theta) = 2 \sin \theta \cos \theta$ [14] $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$ [15]
Divide by $\cos^2 \theta$	$\tan(2\theta) = \frac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta}$
Simplify	$\tan(2\theta) = \frac{\left(\frac{2 \sin \theta \cos \theta}{\cos^2 \theta}\right)}{\left(\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta}\right)}$
Formula	<b><math>\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}</math></b> [18]

## Proof of Half Angle Formulas ( $\theta/2$ )

### Proof

Given	$\cos(2\theta) = 1 - 2 \sin^2 \theta$ [17]
Solve for $\sin^2 \theta$	$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$ [19a]
Substitute	$\theta = \frac{\theta}{2}$
Solve	$\sin^2 \left(\frac{\theta}{2}\right) = \frac{1 - \cos(\theta)}{2}$
Formula	$\sin \left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos(\theta)}{2}}$ [19b]

### Proof

Given	$\cos(2\theta) = 2 \cos^2 \theta - 1$ [16]
Solve for $\cos^2 \theta$	$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$ [20a]
Substitute	$\theta = \frac{\theta}{2}$
Solve	$\cos^2 \left(\frac{\theta}{2}\right) = \frac{1 + \cos(\theta)}{2}$
Formula	$\cos \left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 + \cos(\theta)}{2}}$ [20b]

### Proof

Given	$\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta}$
Substitute	$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$ [19a] $\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$ [20a]
Simplify	$\tan^2 \theta = \frac{\left(\frac{1 - \cos(2\theta)}{2}\right)}{\left(\frac{1 + \cos(2\theta)}{2}\right)} = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$
Substitute	$\theta = \frac{\theta}{2}$
Solve	$\tan^2 \left(\frac{\theta}{2}\right) = \frac{1 - \cos(\theta)}{1 + \cos(\theta)}$ [21a]
Formula	$\tan \left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}$ [21b]

## Proof of Cofunction Formulas

### Proof

Given	$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$ [5]
Substitute	$\alpha = \frac{\pi}{2}, \beta = \theta$
Simplify	$\sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2}\right) \cos \theta + \cos\left(\frac{\pi}{2}\right) \sin \theta$
Formula	$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$ [22]

### Proof

Given	$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ [8]
Substitute	$\alpha = \frac{\pi}{2}, \beta = \theta$
Simplify	$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2}\right) \cos \theta - \sin\left(\frac{\pi}{2}\right) \sin \theta$
Formula	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$ [23]

### Proof

Given	$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
Substitute	$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$ [22] $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$ [23]
Simplify	$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} = \frac{\cos \theta}{\sin \theta} = \cot \theta$
Formula	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$ [24]

### Proof

Given	$\sec(\theta) = \frac{1}{\cos(\theta)}$
Substitute	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$ [23]
Simplify	$\sec\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\cos\left(\frac{\pi}{2} - \theta\right)} = \frac{1}{\sin(\theta)} = \csc \theta$
Formula	$\sec\left(\frac{\pi}{2} - \theta\right) = \csc \theta$ [25]

### Proof

Given	$\csc(\theta) = \frac{1}{\sin(\theta)}$
Substitute	$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta \quad [22]$
Simplify	$\csc\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{1}{\cos(\theta)} = \sec \theta$
Formula	$\csc\left(\frac{\pi}{2} - \theta\right) = \sec \theta \quad [26]$

**Proof**

Given	$\cot(\theta) = \frac{1}{\tan(\theta)}$
Substitute	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta \quad [24]$
Simplify	$\cot\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan\left(\frac{\pi}{2} - \theta\right)} = \frac{1}{\cot(\theta)} = \tan \theta$
Formula	$\cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta \quad [27]$