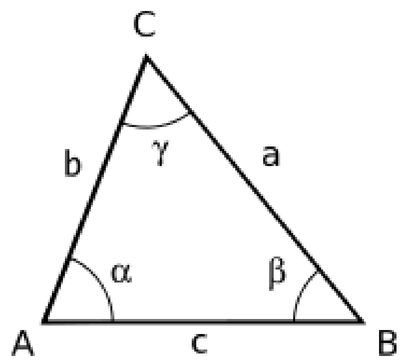
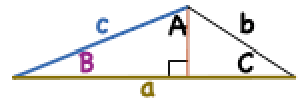
Harold's Triangles Cheat Sheet

21 January 2024

Trig Laws and Formulas

Law	Equation
Reference Triangle	
Law of Sines	$\boldsymbol{\frac{sin(A)}{a} = \frac{sin(B)}{b} = \frac{sin(C)}{c}}$ $\frac{a}{sin(A)} = \frac{b}{sin(B)} = \frac{c}{sin(C)}$
Law of Cosines	$a^2 = b^2 + c^2 - 2bc \cdot cos(A)$ $b^2 = a^2 + c^2 - 2ac \cdot cos(B)$ $\boldsymbol{c^2 = a^2 + b^2 - 2ab \cdot cos(C)}$
Law of Tangents	$\frac{a-b}{a+b} = \frac{tan\left[\frac{(A-B)}{2}\right]}{tan\left[\frac{(A+B)}{2}\right]}$ $\frac{b-c}{b+c} = \frac{tan\left[\frac{(B-C)}{2}\right]}{tan\left[\frac{(B+C)}{2}\right]}$ $\frac{a-c}{a+c} = \frac{tan\left[\frac{(A-C)}{2}\right]}{tan\left[\frac{(A+C)}{2}\right]}$ $tan(A) \cdot tan(B) \cdot tan(C) = tan(A) + tan(B) + tan(C)$
Law of Cotangents	$\frac{cot\left(\frac{A}{2}\right)}{s-a} = \frac{cot\left(\frac{B}{2}\right)}{s-b} = \frac{cot\left(\frac{C}{2}\right)}{s-c}$ $cot(A) \cdot cot(B) \cdot cot(C) = cot(A) + cot(B) + cot(C)$

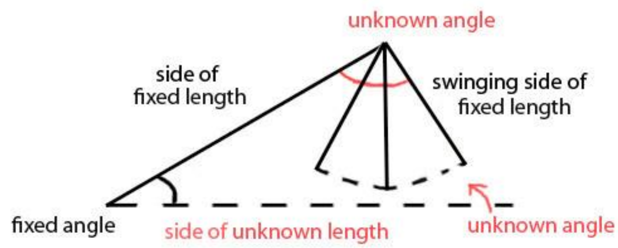
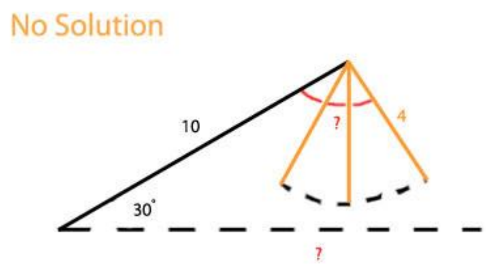
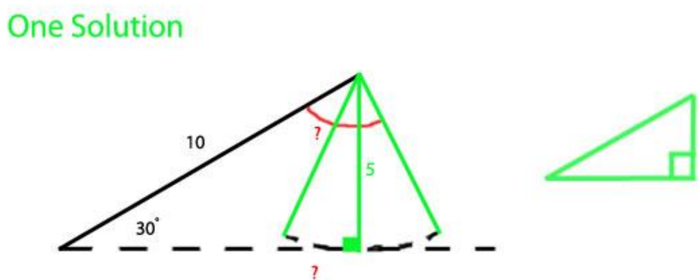
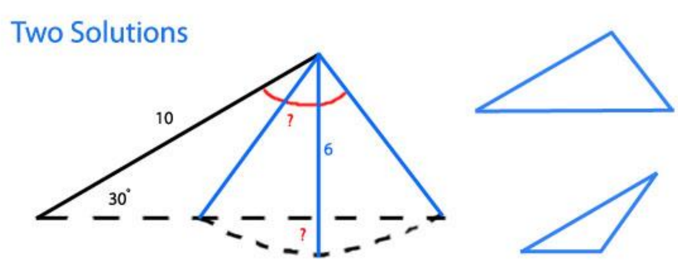
Pythagorean Theorem	If a right triangle, then $$c^2 = a^2 + b^2$$ **Special Case**: Same as Law of Cosines with angle C = 90°. $$c^2 = a^2 + b^2 - 2ab \cdot cos(90°)$$ $$c^2 = a^2 + b^2 \ - 2ab \cdot 0 = a^2 + b^2$$	
Sum of Angles	$A° + B° + C° = 180°$ $C° = 180° - (A° + B°)$	$A + B + C = \pi\ radians$ $C = \pi - (A + B)$
Area Formula	$A = \frac{1}{2}bh = \frac{1}{2}ab \cdot sin(C)$	
Semi-Perimeter	$s = \frac{a + b + c}{2}$	
Heron's Formula	$A = \sqrt{s(s-a)(s-b)(s-c)}$	
Mollweide's Formula	$\frac{a + b}{c} = \frac{cos\left[\frac{(A-B)}{2}\right]}{sin\left(\frac{C}{2}\right)}$	

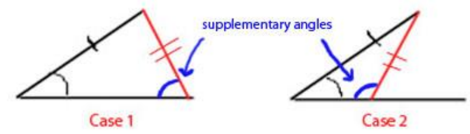
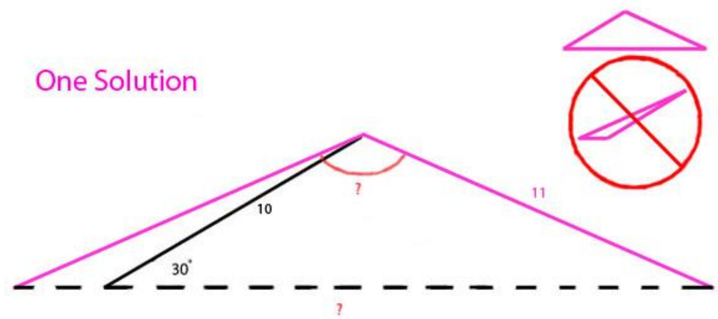
Solving for Angles, Sides, and Area

Order of Use	Comments
1. Sum of Angles	Easiest formula.
2. Pythagorean Theorem	Use if one of the angles is a right angle (90°).
3. Law of Sines	Least complex. Use before Law of Cosines, if possible.
4. Law of Cosines	More complex. Use only once, then use Law of Sines.
5. Heron's Formula	Use for area if all three sides are known.
6. Law of Tangents	Very complex and seldom used.
7. Law of Cotangents	Most complex and seldom used.

<table>
<tr><th rowspan="2">Given</th><th colspan="3">Find</th></tr>
<tr><th>Angle</th><th>Side</th><th>Area</th></tr>
<tr><td>SSS</td><td>Law of Cosines</td><td>Given</td><td>Heron's Formula</td></tr>
<tr><td>SAS</td><td>NA</td><td>Law of Cosines</td><td>$A = \frac{1}{2}ab \cdot sin(C)$</td></tr>
<tr><td>SSA</td><td rowspan="2">Law of Sines</td><td rowspan="2">NA</td><td rowspan="5">$A = \frac{1}{2}bh$</td></tr>
<tr><td>ASS</td></tr>
<tr><td>SAA</td><td rowspan="3">Sum of Angles</td><td rowspan="3">Law of Sines</td></tr>
<tr><td>ASA</td></tr>
<tr><td>AAS</td></tr>
<tr><td>AAA</td><td>Given</td><td colspan="2">Unsolvable. Not unique. One side needed.</td></tr>
</table>

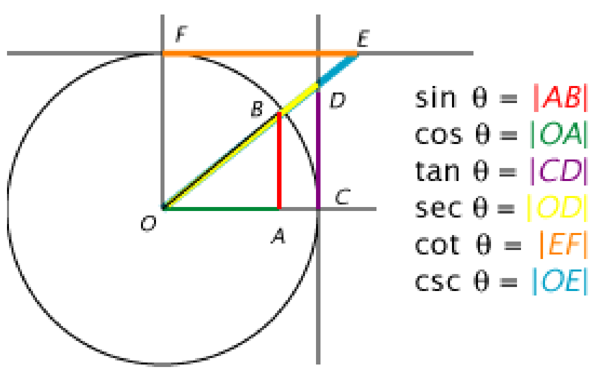
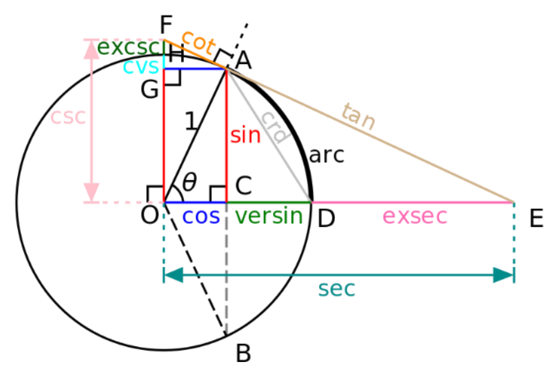
Ambiguous Cases for SSA

Scenario	# of Solutions	Illustrations
SSA	**0 – 2 Solutions**	
a < h	**No Solution**	
a = h	**One Solution**	
b > a > h	**Two Solutions**	

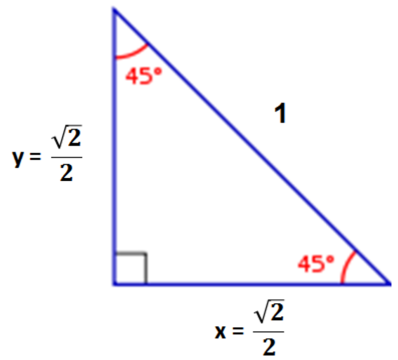
		$\frac{b}{sin\,(B)} = \frac{b}{sin\,(180° - B)}$	
a ≥ b	**One Oblique Solution**		
		When *a* = *b*, equilateral / isososocles When *a* > *b*, obtuse	

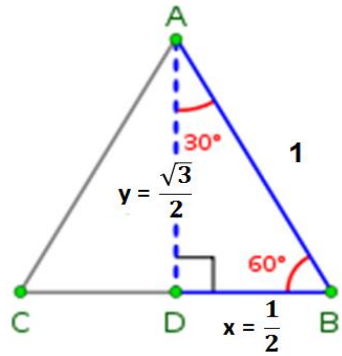
Source: https://mathimages.swarthmore.edu/index.php/Ambiguous_Case

Interesting Trig Lengths on a Unit Circle

Trig Graphically	
	

Fixed Angles Triangles

<table>
<tr><th>45-45-90 Triangle</th><th>30-60-90 Triangle</th></tr>
<tr><td>


</td><td>


</td></tr>
<tr><td>

Proof:

$a^2 + b^2 = c^2$

$x = y$

$x^2 + x^2 = 1^2$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$$\sqrt{x^2} = \sqrt{\frac{1}{2}}$$

$$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

</td><td>

Proof:

$a^2 + b^2 = c^2$

$y^2 + (\frac{1}{2})^2 = 1^2$

$y^2 + \frac{1}{4} = 1$

$y^2 = \frac{3}{4}$

$$\sqrt{y^2} = \sqrt{\frac{3}{4}}$$

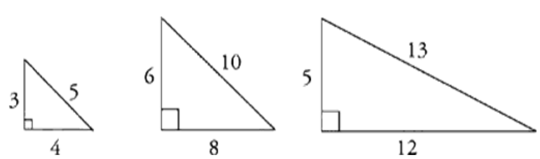
$$y = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{\sqrt{4}} = \pm\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

</td></tr>
</table>

Fixed Sides Triangles

<table>
<tr><th colspan="4">Pythagorean Triples</th></tr>
<tr><td rowspan="15">A Pythagorean triple is a right triangle with only integer sides.

The examples on the right are expressed in the lowest form.

You can scale any of these with an integer (e.g., 2,3,4,5) to generate a family of similar triangles.

</td><td>3 : 4 : 5</td><td>24 : 143 : 145</td><td>84 : 187 : 205</td></tr>
<tr><td>5 : 12 : 13</td><td>28 : 45 : 53</td><td>85 : 132 : 157</td></tr>
<tr><td>8 : 15 : 17</td><td>28 : 195 : 197</td><td>88 : 105 : 137</td></tr>
<tr><td>7 : 24 : 25</td><td>32 : 255 : 257</td><td>95 : 168 : 193</td></tr>
<tr><td>9 : 40 : 41</td><td>33 : 56 : 65</td><td>96 : 247 : 265</td></tr>
<tr><td>11 : 60 : 61</td><td>36 : 77 : 85</td><td>104 : 153 : 185</td></tr>
<tr><td>12 : 35 : 37</td><td>39 : 80 : 89</td><td>105 : 208 : 233</td></tr>
<tr><td>13 : 84 : 85</td><td>44 : 117 : 125</td><td>115 : 252 : 277</td></tr>
<tr><td>15 : 112 : 113</td><td>48 : 55 : 73</td><td>119 : 120 : 169</td></tr>
<tr><td>16 : 63 : 65</td><td>51 : 140 : 149</td><td>120 : 209 : 241</td></tr>
<tr><td>17 : 144 : 145</td><td>52 : 165 : 173</td><td>133 : 156 : 205</td></tr>
<tr><td>19 : 180 : 181</td><td>57 : 176 : 185</td><td>140 : 171 : 221</td></tr>
<tr><td>20 : 21 : 29</td><td>60 : 91 : 109</td><td>160 : 231 : 281</td></tr>
<tr><td>20 : 99 : 101</td><td>60 : 221 : 229</td><td>161 : 240 : 289</td></tr>
<tr><td>21 : 220 : 221</td><td>65 : 72 : 97</td><td>204 : 253 : 325</td></tr>
</table>

Radians and Arc Length	
Radian = arc length (s) of a unit circle s = r θ C = π D = π (2r) = 2πr **Proof:** If r = 1 (unit circle) then s = $(1)\theta$ = θ and C = 2π $(1) = 2\pi$ Therefore 360° = 2π radians To convert degrees to radians: n° x $\left(\frac{\pi \text{ rad}}{180°}\right)$ = m radians To convert radians to degrees: m rad x $\left(\frac{180°}{\pi \text{ rad}}\right)$ = n°