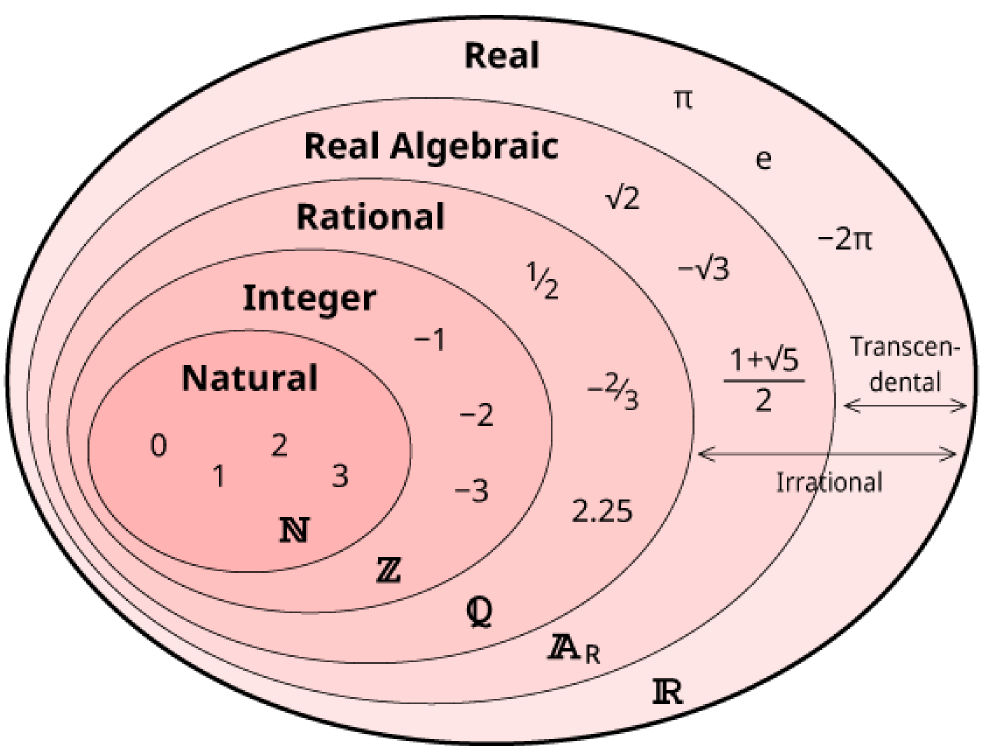
Harold's Real Analysis Cheat Sheet

22 October 2022

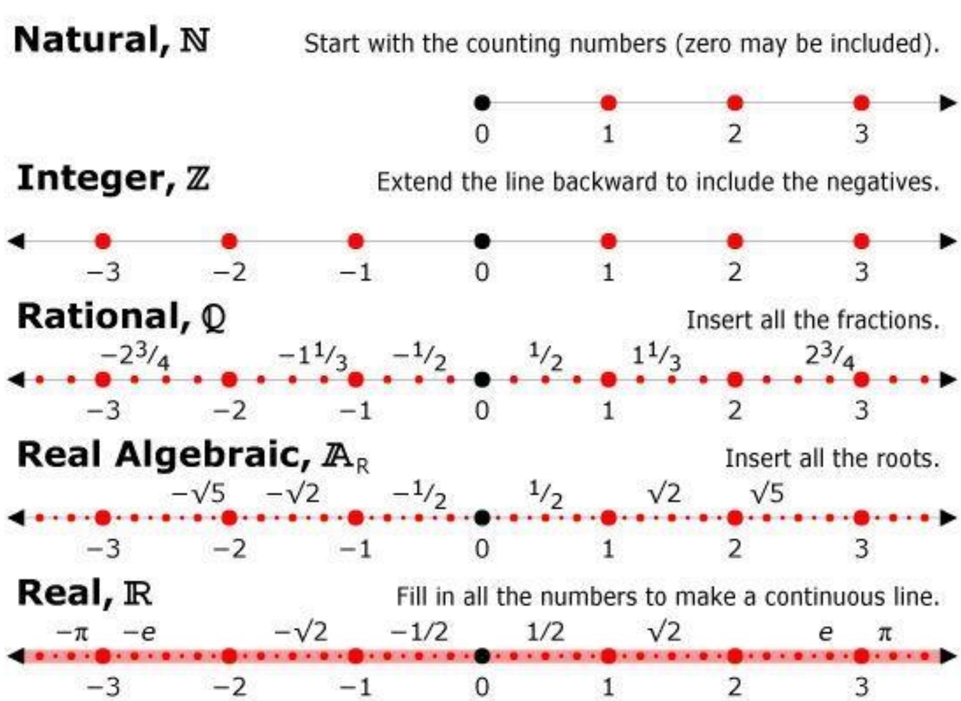
Number Sets

Symbol	Definition	Examples	Equations	Solution
∅	**empty** set, set with no members	{ }	$1 = 2$	null
ℕ	**natural** numbers	$\mathbb{N}_1$ = {1, 2, 3, …}	Pre-2010	NA
		$\mathbb{N}_0$ = {0, 1, 2, 3, …}	See ISO 80000-2 2-6.1	
ℙ	**prime** numbers	{2, 3, 5, 7, 11, 13, …}	unofficial	NA
ℤ	**integers**	{…, −2, −1, 0, 1, 2, …}	$x + 7 = 0$	$x = -7$
ℚ	**rational** numbers	{0, ¼, ½, ¾, 1}	$4x - 1 = 0$	$x = ¼$
𝔸	**algebraic** numbers	{5, -7, ½, $\sqrt{2}$}	$2x^2 + 4x - 7 = 0$	x is algebraic
𝕋	**transcendental** numbers	{π, e, e^π, sin(x), $\log_b$ a}	𝕋 = 𝕌 – 𝔸	NA
ℝ	**real** numbers	{3.1415, -1, ⅞, $\sqrt{2}$, π}	$x^2 - 2 = 0$	$x = \pm\sqrt{2}$
𝕀	**imaginary** numbers	{2i, $\sqrt{-1}$}	$x^2 + 1 = 0$	$x = \pm\sqrt{-1}$ $x = \pm i$
ℂ	**complex** numbers	{1 + 2i, -3.4i, ⅝}	$x^2 - 4x + 5 = 0$	$x = 2 \pm i$
𝕌	**universal** set	{all possible values}	∞	NA



Derived Number Sets

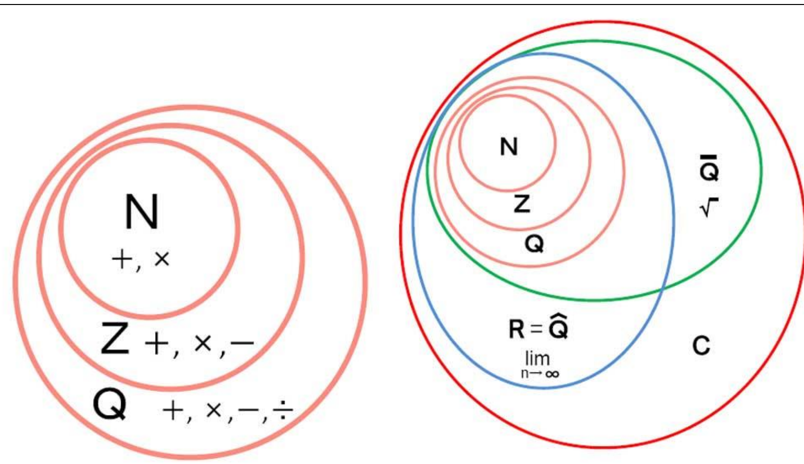
Symbol	Definition	Equations	Examples
Integers $\mathbb{Z}$			
{0}	**zero**	n = 0	{0}
$\mathbb{Z}^*$ $\mathbb{Z}$ - {0} $\mathbb{Z} \setminus$ {0}	**non-zero integers**	n ≠ 0	{-3, -2, -1, 1, 2, 3, …}
$\mathbb{Z}^+$	**positive integers**	n > 0	{1, 2, 3, …}
$\mathbb{N} \cup$ {0}	**non-negative integers**	n ≥ 0	{0, 1, 2, 3, …}
$\mathbb{Z}^-$	**negative integers**	n < 0	{…, -3, -2, -1}
$\mathbb{Z}^- \cup$ {0}	**non-positive integers**	n ≤ 0	{…, -3, -2, -1, 0}
Real Numbers $\mathbb{R}$			
{0}	**zero**	x = 0	{0.0}
$\mathbb{R}$ - {0} $\mathbb{R} \setminus$ {0}	**non-zero real** numbers	x ≠ 0	{-0.001, 0.001}
$\mathbb{R}^+$ (0, ∞)	**positive real** numbers	x > 0	{0.0001, 0.0002, …}
$\mathbb{R}^+ \cup$ {0} [0, ∞)	**non-negative real** numbers	x ≥ 0	{0, 0.0001, 0.0002, …}
$\mathbb{R}^-$ (-∞, 0)	**negative real** numbers	x < 0	{…, -0.0002, -0.0001}
$\mathbb{R}^- \cup$ {0} (-∞, 0]	**non-positive real** numbers	x ≤ 0	{…, -0.0002, -0.0001, 0}



Definitions

Term	Definition
Definition	A precise and unambiguous description of the meaning of a mathematical term. It characterizes the meaning of a word by giving all the properties and only those properties that must be true.
Theorem	A mathematical statement that is proved using rigorous mathematical reasoning. In a mathematical paper, the term theorem is often reserved for the most important results.
Lemma	A minor result whose sole purpose is to help in proving a theorem. It is a steppingstone on the path to proving a theorem. Very occasionally lemmas can take on a life of their own (Zorn's lemma, Urysohn's lemma, Burnside's lemma, Sperner's lemma).
Corollary	A result in which the (usually short) proof relies heavily on a given theorem (we often say that "this is a corollary of Theorem A").
Proposition	A proved and often interesting result, but generally less important than a theorem.
Conjecture	A statement that is unproved, but is believed to be true (Collatz conjecture, Goldbach conjecture, twin prime conjecture).
Claim	An assertion that is then proved. It is often used like an informal lemma.
Axiom / Postulate	A statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proved (Euclid's five postulates, Zermelo-Fraenkel axioms, Peano axioms).
Identity	A mathematical expression giving the equality of two (often variable) quantities (trigonometric identities, Euler's identity).
Paradox	A statement that can be shown, using a given set of axioms and definitions, to be both true and false. Paradoxes are often used to show the inconsistencies in a flawed theory (Russell's paradox). The term paradox is often used informally to describe a surprising or counterintuitive result that follows from a given set of rules (Banach-Tarski paradox, Alabama paradox, Gabriel's horn).

Corollary
↑
Theorem / Proposition
↑
Lemma
↑
Axiom / Postulate
↑
Conjecture / Claim
↑
Definition



Textbook	Bloch, Ethan D.. The Real Numbers and Real Analysis. Springer New York, 2011.

Ch. 1.2: Natural Numbers ℕ

Axiom / Theorem / Lemma / Definition	Description
Operations: Binary, Unary (Definition 1.1.1)	Let S be a set. A **binary** operation on S is a function $S \times S \rightarrow S$. A **unary** operation on S is a function $S \rightarrow S$.
Peano Postulates (Axiom 1.2.1)	There exists a set ℕ with an element $1 \in \mathbb{N}$ and a function $s: \mathbb{N} \rightarrow \mathbb{N}$ that satisfy the following three properties. a. There is no $n \in \mathbb{N}$ such that $s(n) = 1$. b. The function s is injective. c. Let $G \subseteq \mathbb{N}$ be a set. Suppose that $1 \in G$, and that if $g \in G$ then $s(g) \in G$. Then $G = \mathbb{N}$.
Natural Number (Definition 1.2.2)	The set of **natural numbers**, denoted ℕ, is the set the existence of which is given in the Peano Postulates.
Lemma 1.2.3	Let $a \in \mathbb{N}$. Suppose that $a \neq 1$. Then there is a unique $b \in \mathbb{N}$ such that $a = s(b)$.
Definition by Recursion (Theorem 1.2.4)	Let H be a set, let $e \in H$ and let $k: H \rightarrow H$ be a function. Then there is a unique function $f: \mathbb{N} \rightarrow H$ such that $f(1) = e$, and that $f \circ s = k \circ f$.
Operation: + (Theorem 1.2.5)	There is a unique binary operation $+: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ that satisfies the following two properties for all $n,m \in \mathbb{N}$. a. $n + 1 = s(n)$. (successor). b. $n + s(m) = s(n + m)$. $[= n + (m+1)]$
Operation: * (Theorem 1.2.6)	There is a unique binary operation $*: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ that satisfies the following two properties for all $n,m \in \mathbb{N}$. a. $n * 1 = n$. b. $n * s(m) = n(m+1) = (n * m) + n$.
Addition Laws (Theorem 1.2.7a)	Let $a, b, c \in \mathbb{N}$. 1. If $a + c = b + c$, then $a = b$ (Cancellation Law for Addition). 2. $(a + b) + c = a + (b + c)$ (Associative Law for Addition). 3. $1 + a = s(a) = a + 1$. 4. $a + b = b + a$ (Commutative Law for Addition). 5. $a + b \neq 1$. 6. $a + b \neq a$.
Multiplication Laws (Theorem 1.2.7b)	Let $a, b, c \in \mathbb{N}$. 7. $a * 1 = a = 1 * a$ (Identity Law for Multiplication). 8. $(a + b)c = ac + bc$ (Distributive Law). 9. $ab = ba$ (Commutative Law for Multiplication). 10. $c(a + b) = ca + cb$ (Distributive Law). 11. $(ab)c = a(bc)$ (Associative Law for Multiplication). 12. If $ac = bc$ then $a = b$ (Cancellation Law for Multiplication). 13. $ab = 1$ if and only if $a = 1 = b$.
Relation: < (Definition 1.2.8a)	The relation < on ℕ is defined by $a < b$ if and only if there is some $p \in N$ such that $a + p = b$, for all $a,b \in N$.
Relation: ≤ (Definition 1.2.8b)	The relation ≤ on ℕ is defined by $a \leq b$ if and only if $a < b$ or $a = b$, for all $a,b \in \mathbb{N}$.

Relation: $<$ and $\leq$ (Theorem 1.2.9)	Let $a, b, c, d \in \mathbb{N}$. 1. $a \leq a$, and $a \nless a$, and $a < a + 1$. 2. $1 \leq a$. 3. If $a < b$ and $b < c$, then $a < c$; if $a \leq b$ and $b < c$, then $a < c$; if $a < b$ and $b \leq c$, then $a < c$; if $a \leq b$ and $b \leq c$, then $a \leq c$. 4. $a < b$ if and only if $a + c < b + c$. 5. $a < b$ if and only if $ac < bc$. 6. Precisely one of $a < b$ or $a = b$ or $a > b$ holds (Trichotomy Law). 7. $a \leq b$ or $b \leq a$. 8. If $a \leq b$ and $b \leq a$, then $a = b$. 9. It cannot be that $b < a < b + 1$. 10. $a \leq b$ if and only if $a < b + 1$. 11. $a < b$ if and only if $a + 1 \leq b$.
Well-Ordering Principle (Theorem 1.2.10)	Let $G \subseteq \mathbb{N}$ be a non-empty set. Then there is some $m \in G$ such that $m \leq g$ for all $g \in G$.

Ch. 1.3 – 1.4: Integers $\mathbb{Z}$

Axiom, Theorem, etc.	Description
Relation: ~ **(Definition 1.3.1)**	The relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ is defined by $(a,b) \sim (c,d)$ if and only if $a + d = b + c$, for all $(a,b),(c,d) \in \mathbb{N} \times \mathbb{N}$.
Relation: ~ **(Lemma 1.3.2)**	The relation $\sim$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.
Integers: $\mathbb{Z}$ **(Definition 1.3.3)**	The set of integers, denoted $\mathbb{Z}$, is the set of equivalence classes of $\mathbb{N} \times \mathbb{N}$ with respect to the equivalence relation $\sim$.
Well-Defined: +, * **(Lemma 1.3.4)**	The binary operations + and *, the unary operation −, and the relation <, all on $\mathbb{Z}$, are well-defined.
Addition & Multiplication Laws **(Definition 1.4.1 & 1.3.5)**	An **ordered integral domain** is a set R with elements $0,1 \in R$, binary operations + and $\cdot$, a unary operation − and a relation <, which satisfy the following properties. Let $x, y, z \in R$. a. $(x + y) + z = x + (y + z)$ (Associative Law for Addition). b. $x + y = y + x$ (Commutative Law for Addition). c. $x + 0 = x$ (Identity Law for Addition). d. $x + (-x) = 0$ (Inverses Law for Addition). e. $(xy)z = x(yz)$ (Associative Law for Multiplication). f. $xy = yx$ (Commutative Law for Multiplication). g. $x \cdot 1 = x$ (Identity Law for Multiplication). h. $x(y + z) = xy + xz$ (Distributive Law). i. If $xy = 0$, then $x = 0$ or $y = 0$ (No Zero Divisors Law). j. Precisely one of $x < y$ or $x = y$ or $x > y$ holds (Trichotomy Law). k. If $x < y$ and $y < z$, then $x < z$ (Transitive Law). l. If $x < y$ then $x + z < y + z$ (Addition Law for Order). m. If $x < y$ and $z > 0$, then $xz < yz$ (Multiplication Law for Order). n. $0 \neq 1$ (Non-Triviality).
Relation: ≤ **(Definition 1.4.2)**	Let R be an ordered integral domain, and let $A \subseteq R$ be a set. 1. The relation ≤ on R is defined by $a \leq b$ if and only if $a < b$ or $a = b$, for all $a,b \in R$. 2. The set A has a least element if there is some $a \in A$ such that $a \leq x$ for all $x \in A$.
Well-Ordering Principle **(Definition 1.4.3)**	Let R be an ordered integral domain. The ordered integral domain R satisfies the **Well-Ordering Principle** if every non-empty subset of $\{x \in R \mid x > 0\}$ has a least element.
Axiom for the Integers **(Axiom 1.4.4)**	There exists an ordered integral domain $\mathbb{Z}$ that satisfies the Well-Ordering Principle.

Properties of Integers (Lemma 1.4.5 & 1.3.8)	Let $x, y, z \in \mathbb{Z}$. 1. If $x + z = y + z$, then $x = y$ (Cancellation Law for Addition). 2. $-(-x) = x$. 3. $-(x + y) = (-x) + (-y)$. 4. $x \cdot 0 = 0$. 5. If $z \neq 0$ and if $xz = yz$, then $x = y$ (Cancellation Law for Mult.). 6. $(-x)y = -xy = x(-y)$. 7. $xy = 1$ if and only if $x = 1 = y$ or $x = -1 = y$. 8. $x > 0$ if and only if $-x < 0$, and $x < 0$ if and only if $-x > 0$. 9. $0 < 1$. 10. If $x \leq y$ and $y \leq x$, then $x = y$. 11. If $x > 0$ and $y > 0$, then $xy > 0$. If $x > 0$ and $y < 0$, then $xy < 0$.
Discreteness (Theorem 1.4.6 & 1.3.9)	Let $x \in \mathbb{Z}$. Then there is no $y \in \mathbb{Z}$ such that $x < y < x + 1$.
Positive/Negative: +, - (Definition 1.4.7 & 1.3.6)	1. Let $x \in \mathbb{Z}$. The number x is positive if $x > 0$, and the number x is negative if $x < 0$.
$\mathbb{N} \subseteq \mathbb{Z}$: **(Theorem 1.3.7 & Definition 1.4.7)**	Let $i: \mathbb{N} \to \mathbb{Z}$ be defined by $i(n) = [(n+1,1)]$ for all $n \in \mathbb{N}$. 1. The function $i: \mathbb{N} \to \mathbb{Z}$ is injective. 2. $i(\mathbb{N}) = \{x \in \mathbb{Z} \mid x > \hat{0}\}$. 3. $i(1) = \hat{1}$. 4. Let $a,b \in \mathbb{N}$. Then a. $i(a+b) = i(a) + i(b)$; b. $i(ab) = i(a)\, i(b)$; c. $a < b$ if and only if $i(a) < i(b)$.
Natural Numbers: $\mathbb{N}$ **(Definition 1.4.7)**	2. The set of natural numbers, denoted $\mathbb{N}$, is defined by $\mathbb{N} = \{x \in \mathbb{Z} \mid x > 0\}$.
Peano Postulates (Theorem 1.4.8 & Axiom 1.2.1)	Let $s: \mathbb{N} \to \mathbb{N}$ be defined by $s(n) = n + 1$ for all $n \in \mathbb{N}$. a. There is no $n \in \mathbb{N}$ such that $s(n) = 1$. b. The function s is injective. c. Let $G \subseteq \mathbb{N}$ be a set. Suppose that $1 \in G$, and that if $g \in G$ then $s(g) \in G$. Then $G = \mathbb{N}$.

Ch. 1.5: Rational Numbers $\mathbb{Q}$

Definition / Lemma / Theorem	Description
Relation: $\asymp$, $\mathbb{Z}*$ **(Definition 1.5.1)**	Let $\mathbb{Z}* = \mathbb{Z} - \{0\}$. The relation $\asymp$ on $\mathbb{Z} \times \mathbb{Z}*$ is defined by $(x, y) \asymp (z,w)$ if and only if $xw = yz$, for all $(x, y),(z,w) \in \mathbb{Z} \times \mathbb{Z}*$.
Relation: $\asymp$ **(Lemma 1.5.2)**	The relation $\asymp$ is an equivalence relation.
Rational Numbers: $\mathbb{Q}$ **(Definition 1.5.3)**	The set of rational numbers, denoted $\mathbb{Q}$, is the set of equivalence classes of $\mathbb{Z} \times \mathbb{Z}*$ with respect to the equivalence relation $\asymp$. The elements $\bar{0}, \bar{1} \in \mathbb{Q}$ are defined by $\bar{0} = [(0,1)]$ and $\bar{1} = [(1,1)]$. Let $\mathbb{Q}* = \mathbb{Q} - \{\bar{0}\}$. The binary operations + and $\cdot$ on $\mathbb{Q}$ are defined by $[(x,y)] + [(z,w)] = [(xw + yz,yw)]$ $[(x,y)] \cdot [(z,w)] = [(xz, yw)]$ for all $[(x, y)],[(z,w)] \in \mathbb{Q}$. • $-$: The unary operation $-$ on $\mathbb{Q}$ is defined by $-[(x, y)] = [(-x, y)]$ for all $[(x,y)] \in \mathbb{Q}$. • $^{-1}$: The unary operation $^{-1}$ on $\mathbb{Q}*$ is defined by $[(x, y)]^{-1} = [(y, x)]$ for all $[(x, y)] \in \mathbb{Q}*$. • $<$: The relation $<$ on $\mathbb{Q}$ is defined by $[(x,y)] < [(z,w)]$ if and only if either $xw < yz$ when $y > 0$ and $w > 0$ or when $y < 0$ and $w < 0$, • $>$: The relation $>$ on $\mathbb{Q}$ is defined by $[(x,y)] > [(z,w)]$ if and only if either $xw > yz$ when $y > 0$ and $w < 0$ or when $y < 0$ and $w > 0$, for all $[(x, y)],[(z,w)] \in \mathbb{Q}$. • $\leq$: The relation $\leq$ on $\mathbb{Q}$ is defined by $[(x, y)] \leq [(z,w)]$ if and only if $[(x,y)] < [(z,w)]$ or $[(x,y)] = [(z,w)]$, for all $[(x,y)],[(z,w)] \in \mathbb{Q}$.
Well-Defined: $\mathbb{Q}$ **(Lemma 1.5.4)**	The binary operations + and $\cdot$, the unary operations $-$ and $^{-1}$, and the relation $<$, all on $\mathbb{Q}$, are **well-defined**.

Addition and Multiplication Laws (Theorem 1.5.5)	Let $r,s,t \in \mathbb{Q}$. **Field:** 1. $(r + s) + t = r + (s + t)$ (Associative Law for Addition). 2. $r + s = s + r$ (Commutative Law for Addition). 3. $r + \bar{0} = r$ (Identity Law for Addition). 4. $r + (-r) = \bar{0}$ (Inverses Law for Addition). 5. $(rs)t = r(st)$ (Associative Law for Multiplication). 6. $rs = sr$ (Commutative Law for Multiplication). 7. $r \cdot \bar{1} = r$ (Identity Law for Multiplication). 8. If $r \neq \bar{0}$, then $r \cdot r^{-1} = \bar{1}$ (<u>Inverses Law for Multiplication</u>). 9. $r(s + t) = rs + rt$ (Distributive Law). **Ordered Field:** 11. If $r < s$ and $s < t$, then $r < t$ (Transitive Law). 12. If $r < s$ then $r + t < s + t$ (Addition Law for Order). 13. If $r < s$ and $t > \bar{0}$, then $rt < st$ (Multiplication Law for Order). 14. $\bar{0} \neq \bar{1}$ (Non-Triviality).
$\mathbb{Z} \subseteq \mathbb{Q}$: **(Theorem 1.5.6)**	Let $i: \mathbb{Z} \to \mathbb{Q}$ be defined by $i(x) = [(x,1)]$ for all $x \in \mathbb{Z}$. 1. The function $i: \mathbb{Z} \to \mathbb{Q}$ is injective. 2. $i(0) = \bar{0}$ and $i(1) = \bar{1}$. 3. Let $x, y \in \mathbb{Z}$. Then a. $i(x + y) = i(x) + i(y)$; b. $i(-x) = -i(x)$; c. $i(xy) = i(x)\, i(y)$; d. $x < y$ if and only if $i(x) < i(y)$. 4. For each $r \in \mathbb{Q}$ there are $x,y \in \mathbb{Z}$ such that $y \neq 0$ and $r = i(x)\, (i(y))^{-1}$.
Operations: -, $\div$, s^{-1}, $\frac{r}{s}$ **(Definition 1.5.7)**	The binary operation $-$ on $\mathbb{Q}$ is defined by $r - s = r + (-s)$ for all $r,s \in \mathbb{Q}$. The binary operation $\div$ on $\mathbb{Q}*$ is defined by $r \div s = rs^{-1}$ for all $r,s \in \mathbb{Q}*$; we also let $0 \div s = 0 \cdot s^{-1} = 0$ for all $s \in \mathbb{Q}*$. The number $r \div s$ is also denoted $\frac{r}{s}$.
Rational Numbers: $\mathbb{Q}$ **(Lemma 1.5.8)** **(Definition 1.5.3 Restated)**	Let $a,c \in \mathbb{Z}$ and $b,d \in \mathbb{Z}*$. 1. $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. 2. $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. 3. $-\frac{a}{b} = \frac{-a}{b}$. 4. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. 5. If $a \neq 0$, then $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$. 6. If $b > 0$ and $d > 0$, or if $b < 0$ and $d < 0$, then $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$; if $b > 0$ and $d < 0$, or if $b < 0$ and $d > 0$, then $\frac{a}{b} > \frac{c}{d}$ if and only if $ad > bc$.

Ch. 1.6: Dedekind Cuts D_r

Definition / Lemma	Description
Dedekind cut **(Definition 1.6.1)** **AKA "upper cut"**	Let $A \subseteq \mathbb{Q}$ be a set. The set A is a **Dedekind cut** if the following three properties hold. a. $A \neq 0$ and $A \neq \mathbb{Q}$. b. Let $x \in A$. If $y \in \mathbb{Q}$ and $y \geq x$, then $y \in A$. c. Let $x \in A$. Then there is some $y \in A$ such that $y < x$.
Interpreting Dedekind cuts	A Dedekind cut is a set, A, of rational numbers, with the properties shown above. a. Property (a) says A must be nonempty and cannot be all of $\mathbb{Q}$. b. Property (b) says if a number, x, is in A, then all rational numbers greater than x are also in A. c. Property (c) is where things get interesting. It says that if x is in A, then there is at least one element of A that is smaller than x. (Actually, there are infinitely many.) This property is what is going to allow us to fill in the gaps in the rational numbers.
Dedekind cut Existence **(Lemma 1.6.2)**	Let $r \in \mathbb{Q}$. Then the set $\{x \in \mathbb{Q} \mid x > r\}$ is a Dedekind cut.
Dedekind cut not in form of Lemma 1.6.2 **(Example 1.6.3)**	Let $T = \{x \in \mathbb{Q} \mid x > 0 \text{ and } x^2 > 2\}$. (1.6.1) It is seen by Exercise 1.6.2 (1) that T is a Dedekind cut, and by Part (2) of that exercise it is seen that if T has the form $\{x \in \mathbb{Q} \mid x > r\}$ for some $r \in \mathbb{Q}$, then $r^2 = 2$. By **Theorem 2.6.11** we know that there is no rational number x such that $x^2 = 2$, and it follows that T is a Dedekind cut that is not of the form given in Lemma 1.6.2.
Rational cut D_r **(Definition 1.6.4)**	Let $r \in \mathbb{Q}$. The **rational cut** at r, denoted D_r, is the Dedekind cut $D_r = \{x \in \mathbb{Q} \mid x > r\}$. An **irrational cut** is a Dedekind cut that is not a rational cut at any rational number.
Complement of Dedekind cut **(Lemma 1.6.5)**	Let $A \subseteq \mathbb{Q}$ be a Dedekind cut. 1. $\mathbb{Q} - A = \{x \in \mathbb{Q} \mid x < a \text{ for all } a \in A\}$. or $\{ x \in \mathbb{Q} \mid x \leq r \}$. 2. Let $x \in \mathbb{Q} - A$. If $y \in \mathbb{Q}$ and $y \leq x$, then $y \in \mathbb{Q} - A$.
Trichotomy Law **(Lemma 1.6.6)**	Let $A,B \subseteq \mathbb{Q}$ be Dedekind cuts. Then precisely one of $A \subsetneq B$ or $A = B$ or $B \subsetneq A$ holds. **NOTE**: $A \subsetneq B$ means that both $A \subset B$ and $A \neq B$.

Union of Family of Sets (Lemma 1.6.7)	Let A be a non-empty family of subsets of $\mathbb{Q}$. Suppose that X is a Dedekind cut for all X $\in$ A. If $\bigcup_{X \in A} X \neq \mathbb{Q}$, then $\bigcup_{X \in A} X$ is a Dedekind cut. For example, think about what happens if the set A is defined this way: A = {x $\in \mathbb{Q}$ | x > 4}, {x $\in \mathbb{Q}$ | x > 3.2}, {x $\in \mathbb{Q}$ | x > 3.15}, {x $\in \mathbb{Q}$ | x > 3.142}, {x $\in \mathbb{Q}$ | x > 3.1416}, {x $\in \mathbb{Q}$ | x > 3.14160}, {x $\in \mathbb{Q}$ | x > 3.141593}, …} If you were to union all of the elements of A, you would end up with {x $\in \mathbb{Q}$ | x > π}. This is how the "gaps" get filled in.
Dedekind cut Examples (Lemma 1.6.8)	Let A,B ⊆ $\mathbb{Q}$ be Dedekind cuts. 1. The set {r ∈ $\mathbb{Q}$ | r = a + b for some a ∈ A and b ∈ B} is a Dedekind cut. 2. The set {r ∈ $\mathbb{Q}$ | −r < c for some c ∈ $\mathbb{Q}$ − A} is a Dedekind cut. 3. Suppose that 0 ∈ $\mathbb{Q}$ − A and 0 ∈ $\mathbb{Q}$ − B. The set {r ∈ $\mathbb{Q}$ | r = ab for some a ∈ A and b ∈ B} is a Dedekind cut. 4. Suppose that there is some q ∈ $\mathbb{Q}$ − A such that q > 0. The set {r ∈ $\mathbb{Q}$ | r > 0 and $\frac{1}{r}$ < c for some c ∈ $\mathbb{Q}$ − A} is a Dedekind cut.
Well-Ordering Principle (Lemma 1.6.9)	Let A ⊆ $\mathbb{Q}$ be a Dedekind cut. Let y ∈ $\mathbb{Q}$. 1. Suppose that y > 0. Then there are u ∈ A and v ∈ $\mathbb{Q}$ − A such that y = u − v, and v < e for some e ∈ $\mathbb{Q}$ − A. 2. Suppose that y > 1, and that there is some q ∈ $\mathbb{Q}$ − A such that q > 0. Then there are r ∈ A and s ∈ $\mathbb{Q}$ − A such that s > 0, and y > $\frac{r}{s}$, and s < g for some g ∈ $\mathbb{Q}$ − A.

Ch. 1.7: Real Numbers ℝ (Ch. 1)

Axiom / Theorem / Lemma / Definition	Description
Real Numbers: ℝ **Definition 1.7.1**	The set of real numbers, denoted ℝ, is defined by $\mathbb{R} = \{A \subseteq \mathbb{Q} \mid A \text{ is a Dedekind cut}\}$.
Relations: <, ≤ **(Definition 1.7.2)**	The relation < on ℝ is defined by $A < B$ if and only if $A \supsetneqq B$, for all $A,B \in \mathbb{R}$. The relation ≤ on ℝ is defined by $A \leq B$ if and only if $A \supseteq B$, for all $A,B \in \mathbb{R}$.
Operation: +, − **(Definition 1.7.3)**	The binary operation + on ℝ is defined by $A + B = \{r \in \mathbb{Q} \mid r = a + b \text{ for some } a \in A \text{ and } b \in B\}$ for all $A,B \in \mathbb{R}$. The unary operation − on ℝ is defined by $-A = \{r \in \mathbb{Q} \mid -r < c \text{ for some } c \in \mathbb{Q} - A\}$ for all $A \in \mathbb{R}$.
Multiply Operator Setup **Lemma 1.7.4**	Let $A \in \mathbb{R}$, and let $r \in \mathbb{Q}$. 1. $A > D_r$ if and only if there is some $q \in \mathbb{Q} - A$ such that $q > r$. 2. $A \geq D_r$ if and only if $r \in \mathbb{Q} - A$ if and only if $a > r$ for all $a \in A$. 3. If $A < D_0$ then $-A \geq D_0$.
Operations: •, $^{-1}$ **(Definition 1.7.5)**	The binary operation • on ℝ is defined by $A \bullet B = \begin{cases} \{r \in \mathbb{Q} \mid r = ab \text{ for some } a \in A \text{ and } b \in B\}, & \text{if } A \geq D_0 \text{ and } B \geq D_0 \\ -[(-A) \bullet B], & \text{if } A < D_0 \text{ and } B \geq D_0 \\ -[A \bullet (-B)], & \text{if } A \geq D_0 \text{ and } B < D_0 \\ (-A) \bullet (-B), & \text{if } A < D_0 \text{ and } B < D_0. \end{cases}$ The unary operation $^{-1}$ on $\mathbb{R} - \{D_0\}$ is defined by $A^{-1} = \begin{cases} \{r \in \mathbb{Q} \mid r > 0 \text{ and } \frac{1}{r} < c \text{ for some } c \in \mathbb{Q} - A\}, & \text{if } A > D_0 \\ -(-A)^{-1}, & \text{if } A < D_0. \end{cases}$

Addition and Multiplication Laws (Theorem 1.7.6)	Let $A,B,C \in \mathbb{R}$. **Field:** 1. $(A + B) + C = A + (B + C)$ (Associative Law for Addition). 2. $A + B = B + A$ (Commutative Law for Addition). 3. $A + D_0 = A$ (Identity Law for Addition). 4. $A + (-A) = D_0 = 0$ (Inverses Law for Addition). 5. $(AB)C = A(BC)$ (Associative Law for Multiplication). 6. $AB = BA$ (Commutative Law for Multiplication). 7. $A \bullet D_1 = A$ (Identity Law for Multiplication). 8. If $A \neq D_0$, then $AA^{-1} = D_1 = 1$ (Inverses Law for Multiplication). 9. $A(B + C) = AB + AC$ (Distributive Law). **Ordered Field:** 10. Precisely one of $A < B$ or $A = B$ or $A > B$ holds (Trichotomy Law). 11. If $A < B$ and $B < C$, then $A < C$ (Transitive Law). 12. If $A < B$ then $A + C < B + C$ (Addition Law for Order). 13. If $A < B$ and $C > D_0$, then $AC < BC$ (Multiplication Law for Order). 14. $D_0 < D_1$ or $0 < 1$ (Non-Triviality).
Least Upper Bound Property Setup (Definition 1.7.7)	Let $A \subseteq \mathbb{R}$ be a set. 1. The set A is **bounded above** if there is some $M \in \mathbb{R}$ such that $X \leq M$ for all $X \in A$. The number M is called an upper bound of A. 2. The set A is **bounded below** if there is some $P \in \mathbb{R}$ such that $X \geq P$ for all $X \in A$. The number P is called a lower bound of A. 3. The set A is **bounded** if it is bounded above and bounded below. 4. Let $M \in \mathbb{R}$. The number M is a **least upper bound** (also called a **supremum**) of A if M is an upper bound of A, and if $M \leq T$ for all upper bounds T of A. 5. Let $P \in \mathbb{R}$. The number P is a **greatest lower bound** (also called an **infimum**) of A if P is a lower bound of A, and if $P \geq V$ for all lower bounds V of A.
Greatest Lower Bound Property (glb) (Theorem 1.7.8)	Let $A \subseteq \mathbb{R}$ be a set. If A is non-empty and bounded below, then A has a greatest lower bound. (used in Dedekind cut proofs)
Least Upper Bound Property (lub) (Theorem 1.7.9)	Let $A \subseteq \mathbb{R}$ be a set. If A is nonempty and bounded above, then A has a least upper bound.
$\mathbb{Q} \subseteq \mathbb{R}$: **(Theorem 1.7.10)**	Let $i: \mathbb{Q} \to \mathbb{R}$ be defined by $i(r) = D_r$ for all $r \in \mathbb{R}$. 1. The function $i: \mathbb{Q} \to \mathbb{R}$ is injective. 2. $i(0) = D_0$ and $i(1) = D_1$. 3. Let $r,s \in \mathbb{Q}$. Then a. $i(r + s) = i(r) + i(s)$; b. $i(-r) = -i(r)$; c. $i(rs) = i(r)\, i(s)$; d. if $r \neq 0$ then $i(r^{-1}) = [i(r)]^{-1}$; e. $r < s$ if and only if $i(r) < i(s)$.

Ch. 2.2: Real Numbers $\mathbb{R}$

Definitions / Axiom	Description
Addition and Multiplication Laws (Definition 2.2.1)	An ordered field is a set F with elements $0,1 \in F$, binary operations + and $\cdot$, a unary operation −, a relation <, and a unary operation $^{-1}$ on $F - \{0\}$, which satisfy the following properties. Let $x,y,z \in F$. a. $(x + y) + z = x + (y + z)$ (Associative Law for Addition). b. $x + y = y + x$ (Commutative Law for Addition). c. $x + 0 = x$ (Identity Law for Addition). d. $x + (-x) = 0$ (Inverses Law for Addition). e. $(xy)z = x(yz)$ (Associative Law for Multiplication). f. $xy = yx$ (Commutative Law for Multiplication). g. $x \cdot 1 = x$ (Identity Law for Multiplication). h. If $x \neq 0$, then $xx^{-1} = 1$ (Inverses Law for Multiplication). i. $x(y + z) = xy + xz$ (Distributive Law). j. Precisely one of $x < y$ or $x = y$ or $x > y$ holds (Trichotomy Law). k. If $x < y$ and $y < z$, then $x < z$ (Transitive Law). l. If $x < y$ then $x + z < y + z$ (Addition Law for Order). m. If $x < y$ and $z > 0$, then $xz < yz$ (Multiplication Law for Order). n. $0 \neq 1$ (Non-Triviality).
Bounds (Definition 2.2.2)	Let F be an ordered field and let $A \subseteq F$ be a set. 1. The set A is **bounded above** if there is some $M \in F$ such that $x \leq M$ for all $x \in A$. The number M is called an **upper bound** of A. 2. The set A is **bounded below** if there is some $P \in F$ such that $x \geq P$ for all $x \in A$. The number P is called a **lower bound** of A. 3. The set A is **bounded** if it is bounded above and bounded below. 4. Let $M \in F$. The number M is a **least upper bound** (also called a **supremum**) of A if M is an upper bound of A, and if $M \leq T$ for all upper bounds T of A. 5. Let $P \in F$. The number P is a **greatest lower bound** (also called an **infimum**) of A if P is a lower bound of A, and if $P \geq V$ for all lower bounds V of A.
Least Upper Bound Property (Definition 2.2.3)	Let F be an ordered field. The ordered field F satisfies the **Least Upper Bound Property** if every non-empty subset of F that is bounded above has a least upper bound.
Axiom for the Real Numbers (Axiom 2.2.4)	There exists an ordered field $\mathbb{R}$ that satisfies the Least Upper Bound Property.

Ch. 2.3: Algebraic Properties of Real Numbers $\mathbb{R}$

Definitions / Axiom	Description
Operators: −, ÷, ^{2}, ≤, 2 (Definition 2.3.1)	1a. The binary operation − on $\mathbb{R}$ is defined by $a - b = a + (-b)$ for all $a,b \in \mathbb{R}$. 1b. The binary operation ÷ on $\mathbb{R} - \{0\}$ is defined by $a \div b = ab^{-1}$ for all $a,b \in \mathbb{R} - \{0\}$; we also let $0 \div s = 0 \cdot s^{-1} = 0$ for all $s \in \mathbb{R} - \{0\}$. The number $a \div b$ is also denoted $\frac{a}{b}$ or a/b. 2. Let $a \in \mathbb{R}$. The **square** of a, denoted a^2, is defined by $a^2 = a \cdot a$. 3. The relation ≤ on $\mathbb{R}$ is defined by $x \le y$ if and only if $x < y$ or $x = y$, for all $x,y \in \mathbb{R}$. 4. The number $2 \in \mathbb{R}$ is defined by $2 = 1 + 1$.
Properties of Real Numbers (Lemma 2.3.2)	Let $a,b,c \in \mathbb{R}$. 1. If $a + c = b + c$ then $a = b$ (Cancellation Law for Addition). 2. If $a + b = a$ then $b = 0$. 3. If $a + b = 0$ then $b = -a$. 4. $-(a + b) = (-a) + (-b)$. 5. $-0 = 0$. 6. If $ac = bc$ and $c \neq 0$, then $a = b$ (Cancellation Law for Multiplication). 7. $0 \cdot a = 0 = a \cdot 0$. 8. If $ab = a$ and $a \neq 0$, then $b = 1$. 9. If $ab = 1$ then $b = a^{-1}$. 10. If $a \neq 0$ and $b \neq 0$, then $(ab)^{-1} = a^{-1} b^{-1}$. 11. $(-1) \cdot a = -a$. 12. $(-a)b = -ab = a(-b)$. 13. $-(-a) = a$. 14. $(-1)^2 = 1$ and $1^{-1} = 1$. 15. If $ab = 0$, then $a = 0$ or $b = 0$ (No Zero Divisors Law). 16. If $a \neq 0$ then $(a^{-1})^{-1} = a$. 17. If $a \neq 0$ then $(-a)^{-1} = -a^{-1}$.

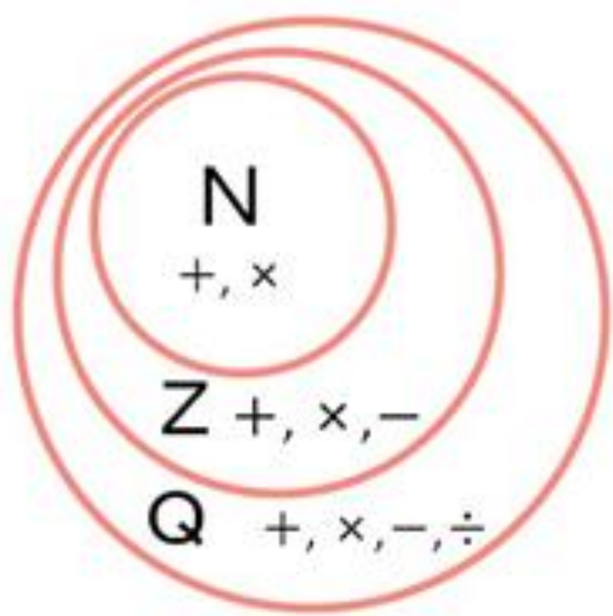
Relations: <, ≤ (Lemma 2.3.3)	Let $a,b,c,d \in \mathbb{R}$. 1. If $a \le b$ and $b \le a$, then $a = b$. 2. If $a \le b$ and $b \le c$, then $a \le c$. If $a \le b$ and $b < c$, then $a < c$. If $a < b$ and $b \le c$, then $a < c$. 3. If $a \le b$ then $a + c \le b + c$. 4. If $a < b$ and $c < d$, then $a + c < b + d$; if $a \le b$ and $c \le d$, then $a + c \le b + d$. 5. $a > 0$ if and only if $-a < 0$, and $a < 0$ if and only if $-a > 0$; also $a \ge 0$ if and only if $-a \le 0$, and $a \le 0$ if and only if $-a \ge 0$. 6. $a < b$ if and only if $b - a > 0$ if and only if $-b < -a$; also $a \le b$ if and only if $b-a \ge 0$ if and only if $-b \le -a$. 7. If $a \ne 0$ then $a^2 > 0$. 8. $-1 < 0 < 1$. 9. $a < a + 1$. 10. If $a \le b$ and $c > 0$, then $ac \le bc$. 11. If $0 \le a < b$ and $0 \le c < d$, then $ac < bd$; if $0 \le a \le b$ and $0 \le c \le d$, then $ac \le bd$. 12. If $a < b$ and $c < 0$, then $ac > bc$. 13. If $a > 0$ then $a^{-1} > 0$. 14. If $a > 0$ and $b > 0$, then $a < b$ if and only if $b^{-1} < a^{-1}$ if and only if $a^2 < b^2$.
Positive / Negative (Definition 2.3.4)	Let $a \in \mathbb{R}$. The number a is **positive** if $a > 0$; the number a is **negative** if $a < 0$; and the number a is **non-negative** if $a \ge 0$.
Positive / Negative (Lemma 2.3.5)	Let $a,b,c,d \in \mathbb{R}$. 1. If $a > 0$ and $b > 0$, then $a + b > 0$. (Addition) If $a > 0$ and $b \ge 0$, then $a + b > 0$. If $a \ge 0$ and $b \ge 0$, then $a + b \ge 0$. 2. If $a < 0$ and $b < 0$, then $a + b < 0$. If $a < 0$ and $b \le 0$, then $a + b < 0$. If $a \le 0$ and $b \le 0$, then $a + b \le 0$. 3. If $a > 0$ and $b > 0$, then $ab > 0$. (Multiplication) If $a > 0$ and $b \ge 0$, then $ab \ge 0$. If $a \ge 0$ and $b \ge 0$, then $ab \ge 0$. 4. If $a < 0$ and $b < 0$, then $ab > 0$. If $a < 0$ and $b \le 0$, then $ab \ge 0$. If $a \le 0$ and $b \le 0$, then $ab \ge 0$. 5. If $a < 0$ and $b > 0$, then $ab < 0$. If $a < 0$ and $b \ge 0$, then $ab \le 0$. If $a \le 0$ and $b > 0$, then $ab \le 0$. If $a \le 0$ and $b \ge 0$, then $ab \le 0$.

Intervals (Definition 2.3.6)	Let $a,b \in \mathbb{R}$. An **open bounded interval** is a set of the form $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$, where $a \le b$. A **closed bounded interval** is a set of the form $[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$, where $a \le b$. A **half-open interval** is a set of the form $[a,b) = \{x \in \mathbb{R} \mid a \le x < b\}$ or $(a,b] = \{x \in \mathbb{R} \mid a < x \le b\}$, where $a \le b$. An **open unbounded interval** is a set of the form $(a,\infty) = \{x \in \mathbb{R} \mid a < x\}$ or $(-\infty,b) = \{x \in \mathbb{R} \mid x < b\}$ or $(-\infty,\infty) = \mathbb{R}$. A **closed unbounded interval** is a set of the form $[a,\infty) = \{x \in \mathbb{R} \mid a \le x\}$ or $(-\infty,b] = \{x \in \mathbb{R} \mid x \le b\}$.
Interval Types	• An **open interval** is either an open bounded interval or an open unbounded interval. • A **closed interval** is either a closed bounded interval or a closed unbounded interval. • A **right unbounded interval** is any interval of the form (a,∞), $[a,\infty)$ or $(-\infty,\infty)$. • A **left unbounded interval** is any interval of the form $(-\infty,b)$, $(-\infty,b]$ or $(-\infty,\infty)$. • A **non-degenerate interval** is any interval of the form (a,b), $(a,b]$, $[a,b)$ or $[a,b]$ where $a < b$, or any unbounded interval. • The number a in intervals of the form $[a,b)$, $[a,b]$ or $[a, \infty)$ is called the **left endpoint** of the interval. • The number b in intervals of the form $(a,b]$, $[a,b]$ or $(-\infty,b]$ is called the **right endpoint** of the interval. • An **endpoint** of an interval is either a left endpoint or a right endpoint. • The **interior** of an interval is everything in the interval other than its endpoints.
Intervals (Lemma 2.3.7)	Let $I \subseteq \mathbb{R}$ be an interval. 1. If $x, y \in I$ and $x \le y$, then $[x, y] \subseteq I$. 2. If I is an open interval, and if $x \in I$, then there is some $\delta > 0$ such that $[x - \delta, x + \delta] \subseteq I$.
Absolute Value (Definition 2.3.8)	Let $a \in \mathbb{R}$. The absolute value of a, denoted $\lvert a\rvert$, is defined by $\lvert a\rvert = (a, \text{ if } a \ge 0 \; -a, \text{ if } a < 0$.
Properties of Absolute Value (Lemma 2.3.9)	Let $a,b \in \mathbb{R}$. 1. $\lvert a\rvert \ge 0$, and $\lvert a\rvert = 0$ if and only if $a = 0$. 2. $-\lvert a\rvert \le a \le \lvert a\rvert$. 3. $\lvert a\rvert = \lvert b\rvert$ if and only if $a = b$ or $a = -b$. 4. $\lvert a\rvert < b$ if and only if $-b < a < b$, and $\lvert a\rvert \le b$ if and only if $-b \le a \le b$. 5. $\lvert ab\rvert = \lvert a\rvert \cdot \lvert b\rvert$. 6. $\lvert a + b\rvert \le \lvert a\rvert + \lvert b\rvert$ (Triangle Inequality). 7. $\lvert \lvert a\rvert - \lvert b\rvert \rvert \le \lvert a + b\rvert$ and $\lvert \lvert a\rvert - \lvert b\rvert \rvert \le \lvert a - b\rvert$.
Epsilon: ε ≈ 0 (Lemma 2.3.10)	Let $a \in \mathbb{R}$. 1. $a \le 0$ if and only if $a < \varepsilon$ for all $\varepsilon > 0$. 2. $a \ge 0$ if and only if $a > -\varepsilon$ for all $\varepsilon > 0$. 3. $a = 0$ if and only if $\lvert a\rvert < \varepsilon$ for all $\varepsilon > 0$.

2.4 Real Numbers Include Natural, Integers, and Rationals ($\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$)

Theorem / Lemma / Definition / Corollary	Description
Inductive Set (Definition 2.4.1)	Let $S \subseteq \mathbb{R}$ be a set. The set S is **inductive** if it satisfies the following two properties. (a) $1 \in S$. (b) If $a \in S$, then $a + 1 \in S$.
Definition: $\mathbb{N}$ (Definition 2.4.2)	The set of **natural numbers**, denoted $\mathbb{N}$, is the intersection of all inductive subsets of $\mathbb{R}$.
Properties of $\mathbb{N}$ (Lemma 2.4.3)	1. $\mathbb{N}$ is inductive. 2. If $A \subseteq \mathbb{R}$ and A is inductive, then $\mathbb{N} \subseteq A$. 3. If $n \in \mathbb{N}$ then $n \geq 1$.
Peano Postulates (Theorem 2.4.4)	Let $s: N \to N$ be defined by $s(n) = n + 1$ for all $n \in \mathbb{N}$. a. There is no $n \in \mathbb{N}$ such that $s(n) = 1$. b. The function s is injective. c. Let $G \subseteq \mathbb{N}$ be a set. Suppose that $1 \in G$, and that if $g \in G$ then $s(g) \in G$. Then $G = \mathbb{N}$.
$\mathbb{N}$ Closed Under +, · (Lemma 2.4.5)	Let $a,b \in \mathbb{N}$. Then $a + b \in \mathbb{N}$ and $ab \in \mathbb{N}$.
Well-Ordering Principle (Theorem 2.4.6)	Let $G \subseteq \mathbb{N}$ be a non-empty set. Then there is some $m \in G$ such that $m \leq g$ for all $g \in G$.
Definition: $\mathbb{Z}$ (Definition 2.4.7)	Let $-\mathbb{N} = \{x \in \mathbb{R} \mid x = -n \text{ for some } n \in \mathbb{N}\}$. The set of **integers**, denoted $\mathbb{Z}$, is defined by $\mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}$.
Properties of $\mathbb{Z}$ (Lemma 2.4.8)	1. $\mathbb{N} \subseteq \mathbb{Z}$. 2. $a \in \mathbb{N}$ if and only if $a \in \mathbb{Z}$ and $a > 0$. 3. The three sets $-\mathbb{N}$, $\{0\}$ and $\mathbb{N}$ are mutually disjoint.
$\mathbb{Z}$ Closed Under +, ·, − (Lemma 2.4.9)	Let $a,b \in \mathbb{Z}$. Then $a + b \in \mathbb{Z}$, and $ab \in \mathbb{Z}$, and $-a \in \mathbb{Z}$.
Integers are Discrete (Theorem 2.4.10)	Let $a,b \in \mathbb{Z}$. 1. If $a < b$ then $a + 1 \leq b$. 2. There is no $c \in \mathbb{Z}$ such that $a < c < a + 1$. 3. If $\lvert a-b \rvert < 1$ then $a = b$.
Definition: $\mathbb{Q}$ (Definition 2.4.11)	The set of **rational numbers**, denoted $\mathbb{Q}$, is defined by $\mathbb{Q} = \{x \in \mathbb{R} \mid x = a / b \text{ for some } a,b \in \mathbb{Z} \text{ such that } b \neq 0\}$. The set of **irrational numbers** is the set $\mathbb{R} - \mathbb{Q}$.
Properties of $\mathbb{Q}$ (Lemma 2.4.12)	1. $\mathbb{Z} \subseteq \mathbb{Q}$. 2. $q \in \mathbb{Q}$ and $q > 0$ if and only if $q = a / b$ for some $a,b \in \mathbb{N}$.

Fraction Manipulation (Lemma 2.4.13)	Let a,b,c,d ∈ ℤ. Suppose that b ≠ 0 and d ≠ 0. 1. a / b = 0 if and only if a = 0. 2. a / b = 1 if and only if a = b. 3. a / b = c / d if and only if ad = bc. 4. a / b + c / d = (ad + bc) / bd. 5. −(a / b) = (−a) / b = a / (−b). 6. a / b · c / d = ac / bd. 7. If a ≠ 0, then $(a / b)^{-1}$ = b / a.
ℚ **Closed Under +,** ·, −, $^{-1}$ **(Corollary 2.4.14)**	Let a,b ∈ ℚ. Then a + b ∈ ℚ, and ab ∈ ℚ, and −a ∈ ℚ, and if a ≠ 0 then a^{-1} ∈ ℚ.



Ch. 2.5: Induction and Recursion

Proposition / Theorem / Lemma / Definition	Description
Principle of Mathematical Induction (Theorem 2.5.1)	Let $G \subseteq \mathbb{N}$. Suppose that a. $1 \in G$; b. if $n \in G$, then $n + 1 \in G$. Then $G = \mathbb{N}$.
Proposition 2.5.2	Example induction proof
Definition 2.5.3	Let $a,b \in \mathbb{Z}$. The set $\{a, \ldots, b\}$ is defined by $\{a, \ldots, b\} = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$.
Principle of Mathematical Induction—Variant/Complete (Theorem 2.5.4)	Let $G \subseteq \mathbb{N}$. Suppose that a. $1 \in G$; b. if $n \in \mathbb{N}$ and $\{1, \ldots, n\} \subseteq G$, then $n + 1 \in G$. Then $G = \mathbb{N}$.
Definition by Recursion (Theorem 2.5.5)	Let H be a set, let $e \in H$ and let $k: H \to H$ be a function. Then there is a unique function $f: \mathbb{N} \to H$ such that $f(1) = e$, and that $f(n + 1) = k(f(n))$ for all $n \in \mathbb{N}$.
Definition of x^n Definition 2.5.6	Let $x \in \mathbb{R}$. The number $x^n \in \mathbb{R}$ is defined for all $n \in \mathbb{N}$ by letting $x^1 = x$, and $x^{n+1} = x \cdot x^n$ for all $x \in \mathbb{N}$.
Lemma 2.5.7	Let $x \in \mathbb{R}$. Suppose that $x \neq 0$. Then $x^n \neq 0$ for all $n \in \mathbb{N}$.
Definition: x^0 Definition 2.5.8	Let $x \in \mathbb{R}$. Suppose that $x \neq 0$. The number $x^0 \in \mathbb{R}$ is defined by $x^0 = 1$. For each $n \in \mathbb{N}$, the number x^{-n} is defined by $x^{-n} = (x^n)^{-1}$.
Power Rules Lemma 2.5.9	Let $x \in \mathbb{R}$, and let $n,m \in \mathbb{Z}$. Suppose that $x \neq 0$. 1. $x^n x^m = x^{n+m}$. 2. $x^n / x^m = x^{n-m}$.
Polynomial Function Definition 2.5.10	Let $A \subseteq \mathbb{R}$ be a set, and let $f: A \to \mathbb{R}$ be a function. The function f is a **polynomial function** if there are some $n \in \mathbb{N} \cup \{0\}$ and $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ for all $x \in A$.
$a_{n+1} = n + a_n$ Theorem 2.5.11	Let H be a set, let $e \in H$ and let $t: H \times \mathbb{N} \to H$ be a function. Then there is a unique function $g: \mathbb{N} \to H$ such that $g(1) = e$, and that $g(n + 1) = t((g(n), n))$ for all $n \in \mathbb{N}$.
Factorial: n! Example 2.5.12	We want to define a sequence of real numbers $a_1, a_2, a_3, \ldots$ such that $a_1 = 1$, and $a_{n+1} = (n + 1)a_n$ for all $n \in \mathbb{N}$.
max() Function (Example 2.5.13)	$max\{x, y\} = \begin{cases} x, & if\ x \geq y \\ y, & if\ x \leq y \end{cases}$
Exercise 2.5.3	Let $n \in \mathbb{N}$, and let $a_1, a_2, \ldots, a_n \in \mathbb{R}$. Prove that $\lvert a_1 + a_2 + \cdot\cdot\cdot + a_n \rvert \leq \lvert a_1 \rvert + \lvert a_2 \rvert + \cdot\cdot\cdot + \lvert a_n \rvert$.

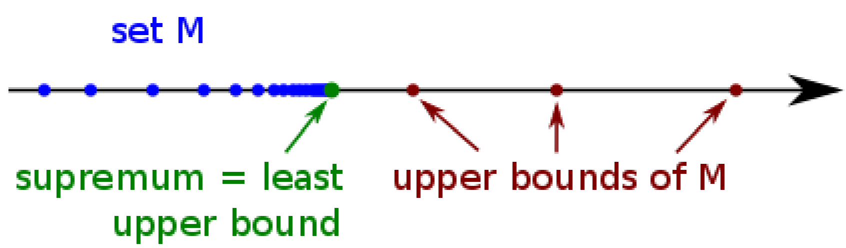
Ch. 2.6: The Least Upper Bound Property

Theorem / Lemma / Corollary / Definition	Description
Example 2.6.1	(1) Let $A = [3,5)$. Then 10 is an upper bound of A, and -100 is a lower bound. Hence A is bounded above and bounded below, and therefore A is bounded.
Unique LUB / GLB (Lemma 2.6.2)	Let $A \subseteq \mathbb{R}$ be a non-empty set. 1. If A has a least upper bound, the least upper bound is unique. 2. If A has a greatest lower bound, the greatest lower bound is unique.
lub A / glb A (Definition 2.6.3)	Let $A \subseteq \mathbb{R}$ be a non-empty set. If A has a least upper bound, it is denoted lub A. If A has a greatest lower bound, it is denoted glb A.
Least Upper Bound Property (Theorem 1.7.9)	Let $A \subseteq \mathbb{R}$ be a set. If A is nonempty and bounded above, then A has a least upper bound.
Greatest Lower Bound Property (Theorem 2.6.4)	Let $A \subseteq \mathbb{R}$ be a set. If A is non-empty and bounded below, then A has a greatest lower bound.
Lemma 2.6.5	Let $A \subseteq \mathbb{R}$ be a non-empty set, and let $\varepsilon > 0$. 1. Suppose that A has a least upper bound. Then there is some $a \in A$ such that $\text{lub } A - \varepsilon < a \le \text{lub } A$. 2. Suppose that A has a greatest lower bound. Then there is some $b \in A$ such that $\text{glb } A \le b < \text{glb } A + \varepsilon$.
No Gap Lemma (Lemma 2.6.6)	Let $A,B \subseteq \mathbb{R}$ be non-empty sets. Suppose that if $a \in A$ and $b \in B$, then $a \le b$. 1. A has a least upper bound and B has a greatest lower bound, and $\text{lub } A \le \text{glb } B$. 2. $\text{lub } A = \text{glb } B$ if and only if for each $\varepsilon > 0$, there are $a \in A$ and $b \in B$ such that $b - a < \varepsilon$.
Archimedean Property (Theorem 2.6.7)	Let $a,b \in \mathbb{R}$. Suppose that $a > 0$. Then there is some $n \in \mathbb{N}$ such that $b < na$.
$\mathbb{R}$ In-between $\mathbb{Z}$s (Corollary 2.6.8)	Let $x \in \mathbb{R}$. 1. There is a unique $n \in \mathbb{Z}$ such that $n - 1 \le x < n$. If $x \ge 0$, then $n \in \mathbb{N}$. 2. If $x > 0$, there is some $m \in \mathbb{N}$ such that $1 / m < x$.
Square Root Theorem 2.6.9	Let $p \in (0,\infty)$. Then there is a unique $x \in (0,\infty)$ such that $x^2 = p$.
Square Root: $\sqrt{\ }$ Definition 2.6.10	Let $p \in (0,\infty)$. The square root of p, denoted $\sqrt{p}$, is the unique $x \in (0,\infty)$ such that $x^2 = p$.
$\sqrt{2}$ is Irrational (Theorem 2.6.11)	Let $p \in \mathbb{N}$. Suppose that there is no $u \in \mathbb{Z}$ such that $p = u^2$. Then $\sqrt{p} \notin \mathbb{Q}$.

$\mathbb{Q}$ ≠ LUB (Corollary 2.6.12)	The ordered field $\mathbb{Q}$ does not satisfy the Least Upper Bound Property.
$\mathbb{R}$ Sandwich (Theorem 2.6.13)	Let a,b ∈ $\mathbb{R}$. Suppose that a < b. 1. There is some q ∈ $\mathbb{Q}$ such that a < q < b. 2. There is some r ∈ $\mathbb{R} - \mathbb{Q}$ such that a < r < b.
Heine–Borel Theorem (Theorem 2.6.14)	Let C ⊆ $\mathbb{R}$ be a closed bounded interval, let I be a non-empty set and let $\{A_i\}_{i \in \mathrm{I}}$ be a family of open intervals in $\mathbb{R}$. Suppose that $C \subseteq \bigcup_{i \in \mathrm{I}} A_i$. Then there are n ∈ $\mathbb{N}$ and $i_1, i_2, \ldots, i_n$ ∈ I such that $C \subseteq \bigcup_{k=1}^{n} A_{i_k}$.

Ch. 2.7: Uniqueness of the Real Numbers

Theorem	Description
Uniqueness of the Real Numbers (Theorem 2.7.1)	Let R_1 and R_2 be ordered fields that satisfy the Least Upper Bound Property. Then there is a function f: $R_1 \rightarrow R_2$ that is bijective, and that satisfies the following properties. Let x,y ∈ R_1. a. f(x + y) = f(x) + f(y). b. f(xy) = f(x) f(y). c. If x < y, then f(x) < f(y).



Ch. 2.8: Decimal Expansion of Real Numbers

Theorem / Lemma / Definition	Description
Base-p (Lemma 2.8.1)	Let p ∈ ℕ. Suppose that p > 1. Let n ∈ ℕ. Then there is a unique k ∈ ℕ such that $p^{k-1} \le n < p^k$.
Base-p Numbers (Theorem 2.8.2)	Let p ∈ ℕ. Suppose that p > 1. Let n ∈ ℕ. Then there are unique k ∈ ℕ and $a_0, a_1, \ldots, a_{k-1} \in \{0, \ldots, p-1\}$ such that $a_{k-1} \neq 0$, and that $$n = \sum_{i=0}^{k-1} a_i p^i.$$
Base-p Fractions (Lemma 2.8.3)	Let p ∈ ℕ. Suppose that p > 1. Let $a_1, a_2, a_3, \ldots \in \{0, \ldots, p-1\}$. Then the set $$\left\{\sum_{i=1}^{n} a_i p^{-i} \mid n \in \mathbb{N}\right\}$$ is bounded below by 0 and is bounded above by 1. [0,1]
Definition 2.8.4	Let p ∈ ℕ. Suppose that p > 1. Let $a_1, a_2, a_3, \ldots \in \{0, \ldots, p-1\}$. The sum $\sum_{i=1}^{\infty} a_i p^{-i}$ is defined by $$\sum_{i=1}^{\infty} a_i p^{-i} = lub\left\{\sum_{i=1}^{n} a_i p^{-i} \mid n \in \mathbb{N}\right\}$$
Lemma 2.8.5	Let p ∈ ℕ. Suppose that p > 1. Let $a_1, a_2, a_3, \ldots \in \{0, \ldots, p-1\}$. 1. $0 \le \sum_{i=1}^{\infty} a_i p^{-i} \le 1$. 2. $\sum_{i=1}^{\infty} a_i p^{-i} = 0$ if and only if $a_i = 0$ for all i ∈ ℕ. 3. $\sum_{i=1}^{\infty} a_i p^{-i} = 1$ if and only if $a_i = p - 1$ for all i ∈ ℕ. 4. Let m ∈ ℕ. Suppose that m > 1, and that $a_{m-1} \neq p - 1$. Then $$\sum_{i=1}^{\infty} a_i p^{-i} \le \sum_{i=1}^{m-2} a_i p^{-i} + \frac{a_{m-1}+1}{p^{m-1}},$$ where equality holds if and only if $a_i = p - 1$ for all i ∈ ℕ such that i ≥ m.

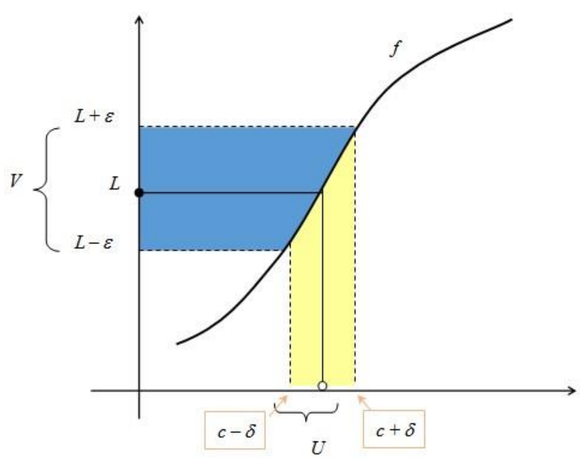
Uniqueness of $\mathbb{R}$ (Theorem 2.8.6)	Let $p \in \mathbb{N}$. Suppose that $p > 1$. Let $x \in (0, \infty)$. 1. There are $k \in \mathbb{N}$, and $b_0, b_1, ..., b_{k-1} \in \{0, ..., p-1\}$ and $a_1, a_2, a_3 ... \in \{0, ..., p-1\}$, such that $$x = \sum_{j=0}^{k-1} b_j p^j + \sum_{i=1}^{\infty} a_i p^{-i}.$$ 2. It is possible to choose $k \in \mathbb{N}$, and $b_0, b_1, ..., b_{k-1} \in \{0, ..., p-1\}$, and $a_1, a_2, a_3 ... \in \{0, ..., p-1\}$ in Part (1) of this theorem such that there is no $m \in \mathbb{N}$ such that $a_i = p-1$ for all $i \in \mathbb{N}$ such that $i \geq m$. 3. If $x > 1$, then it is possible to choose $k \in \mathbb{N}$, and $b_0, b_1, ..., b_{k-1} \in \{0, ..., p-1\}$, and $a_1, a_2, a_3 ... \in \{0, ..., p-1\}$ in Part (1) of this theorem such that $b_{k-1} \neq 0$. If $0 < x < 1$, then it is possible to choose $k = 1$, and $b_0 = 0$, and $a_1, a_2, a_3 ... \in \{0, ..., p-1\}$ in Part (1) of this theorem. 4. If the conditions of Parts (2) and (3) of this theorem hold, then the numbers $k \in \mathbb{N}$, and $b_0, b_1, ..., b_{k-1} \in \{0, ..., p-1\}$, and $a_1, a_2, a_3 ... \in \{0, ..., p-1\}$ in Part (1) are **unique**.
Base p Representation ($b_j.a_i$) (Definition 2.8.7)	Let $p \in \mathbb{N}$. Suppose that $p > 1$. Let $x \in (0, \infty)$. A **base p representation** of the number x is an expression of the form $x = b_{k-1} \cdots b_1 b_0.a_1 a_2 a_3 \cdots$, where $k \in \mathbb{N}$ and $b_0, b_1, ..., b_{k-1} \in \{0, ..., p-1\}$ and $a_1, a_2, a_3 ... \in \{0, ..., p-1\}$ are such that $$x = \sum_{j=0}^{k-1} b_j p^j + \sum_{i=1}^{\infty} a_i p^{-i}.$$
Division Algorithm: ÷ (Theorem 2.8.8)	Let $a \in \mathbb{N} \cup \{0\}$ and $b \in \mathbb{N}$. Then there are unique $q,r \in \mathbb{N} \cup \{0\}$ such that $a = bq + r$ and $0 \leq r < b$. (q = quotient, r = remainder)
Repeating Decimal (Definition 2.8.9)	Let $p \in \mathbb{N}$. Suppose that $p > 1$. Let $x \in (0, \infty)$, and let $x = b_{k-1} \cdots b_1 b_0.a_1 a_2 a_3 \cdots$ be a base p representation of x. This base p representation is **eventually repeating** if there are some $r,s \in \mathbb{N}$ such that $a_j = a_{j+s}$ for all $j \in \mathbb{N}$ such that $j \geq r$; in that case we write $$x = b_{k-1} \cdots b_1 b_0.a_1 a_2 a_3 \cdots a_{r-1} \overline{a_r \cdots a_{r+s-1}}.$$
Rational if Repeating Decimal (Theorem 2.8.10)	Let $p \in \mathbb{N}$. Suppose that $p > 1$. Let $x \in (0, \infty)$. Then $x \in \mathbb{Q}$ if and only if x has an eventually repeating base p representation.

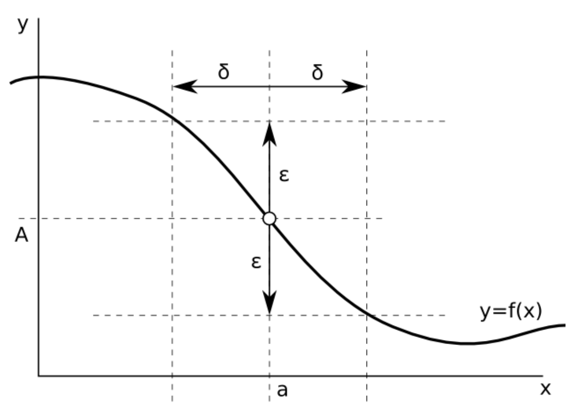
Ch. 3.2 Limits of Functions

Theorem / Lemma / Definition	Description
Limit of a Function (Definition 3.2.1)	Let I ⊆ ℝ be an open interval, let c ∈ I, let f: I − {c} → ℝ be a function and let L ∈ ℝ. The number L is the **limit** of f as x goes to c, written $$\lim_{x \to c} f(x) = L$$ if for each ε > 0, there is some δ > 0 such that x ∈ I − {c} and 0 < |x − c| < δ imply |f(x) − L| < ε. If $\lim_{x \to c} f(x) = L$, we also say that f **converges** to L as x goes to c. If f converges to some real number as x goes to c, we say that $\lim_{x \to c} f(x)$ exists. An **open interval** is an interval that does not include its end points.
Logical Form of Limits	(∀ε > 0) (∃δ > 0) [(x ∈ I − {c} ∧ |x − c| < δ) → |f(x) − L| < ε] The order of the quantifiers in the definition of limits is absolutely crucial.
Proof Format	A typical proof that $\lim_{x \to c} f(x) = L$ must therefore have the following form: **Proof**. Let ε > 0 . . . . (argumentation) . . . Let δ = f(ε) . . . (argumentation) . . . Suppose that x ∈ I − {c} and |x − c| < δ . . . . (argumentation) . . . Therefore |f(x) − L| < ε.
L is Unique (Lemma 3.2.2)	Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I − {c} → ℝ be a function. If $\lim_{x \to c} f(x) = L$ for some L ∈ ℝ, then L is unique.
Example Proofs (Example 3.2.3)	(1) Prove that $\lim_{x \to 4}(5x + 1) = 21$. **Proof:** Let ε > 0. Let δ = ε 5. Suppose that x ∈ ℝ − {4} and |x−4| < δ. Then |(5x + 1) − 21| = |5x − 20| = 5|x − 4| < 5δ = 5 · ε 5 = ε.
	(2) Prove that $\lim_{x \to 3}(x^2 - 1) = 8$. **Proof:** Let ε > 0. Let δ = min{ε 7 ,1}. Suppose that x ∈ℝ − {3} and |x−3| < δ. **Then** |x−3| < 1, which **implies** that −1 < x−3 < 1, and **therefore** 2 < x < 4, and **hence** 5 < x + 3 < 7, and we **conclude** that 5 < |x + 3| < 7. Then |(x2 − 1) − 8| = |x2 − 9| = |x − 3|·|x + 3| < δ · 7 ≤ ε 7 · 7 = ε.

	(3) Prove that $\lim_{x\to 0}\left(\frac{1}{x}\right)$. does not exist. **Proof:** Suppose that $\lim_{x\to 0}\left(\frac{1}{x}\right) = L$ for some L ∈ ℝ. Let ε = |L| / 2 if L ≠ 0, and let ε = 1 if L = 0. We consider the case when L > 0; the other cases are similar. Let δ > 0. Because L > 0, then L + ε > 0. Let x = min{ δ /2, 1 / (L + ε)}. Then x ∈ (0, ∞) and |x − 0| ≤ δ / 2 < δ. On the other hand, because x ≤ 1 / (L + ε), it follows that L + ε ≤ 1 / x, and hence 1 / x − L ≥ ε, which implies that |1 / x − L| ≮ ε.
Sign-Preserving Property for Limits (Theorem 3.2.4)	Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I − {c} → ℝ be a function. Suppose that $\lim_{x\to c} f(x)$ exists. 1. If $\lim_{x\to c} f(x) > 0$, then there is some M > 0 and some δ > 0 such that x ∈ I − {c} and |x − c| < δ imply f(x) > M. 2. If $\lim_{x\to c} f(x) < 0$, then there is some N < 0 and some δ > 0 such that x ∈ I − {c} and |x − c| < δ imply f(x) < N.
Bounded (Lemma 3.2.7)	Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I − {c} → ℝ be a function. If $\lim_{x\to c} f(x)$ exists, then there is some δ > 0 such that the restriction of f to (I − {c}) ∩ (c − δ, c + δ) is bounded.
Zero (Lemma 3.2.8)	Let I ⊆ ℝ be an open interval, let c ∈ I and let f,g: I − {c} → ℝ be functions. Suppose that $\lim_{x\to c} f(x) = 0$, and that g is bounded. Then $\lim_{x\to c} f(x)\, g(x) = 0$.
Functions for +, -, k, •, ÷ (Definition 3.2.9)	Let A,B be sets, let f: A → ℝ and g: B → ℝ be functions and let k ∈ ℝ. 1. The function f + g: A ∩ B → ℝ is defined by [f + g](x) = f(x) + g(x) for all x ∈ A ∩ B. 2. The function f − g: A ∩ B → ℝ is defined by [f − g](x) = f(x) − g(x) for all x ∈ A ∩ B. 3. The function k f: A → ℝ is defined by [k f](x) = k f(x) for all x ∈ A. 4. The function f · g: A ∩ B → ℝ is defined by [f · g](x) = f(x) · g(x) for all x ∈ A ∩ B. 5. Let C = (A ∩ B) − {b ∈ B | g(b) = 0}. The function f g: C → ℝ is defined by [f/g] (x) = f(x) / g(x) for all x ∈ C. 6. The function | f |: A → ℝ is defined by | f |(x) = | f(x) | for all x ∈ A.

Limits for +, -, k, •, ÷ (Theorem 3.2.10)	Let I ⊆ ℝ be an open interval, let c ∈ I, let f,g: I − {c} → ℝ be functions and let k ∈ ℝ. Suppose that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. 1. $\lim_{x\to c}[f+g](x)$ exists and $\lim_{x\to c}[f+g](x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$. 2. $\lim_{x\to c}[f-g](x)$ exists and $\lim_{x\to c}[f-g](x) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$. 3. $\lim_{x\to c}[k\cdot f](x)$ exists and $\lim_{x\to c}[k\cdot f](x) = k\cdot\lim_{x\to c} f(x)$. 4. $\lim_{x\to c}[f\cdot g](x)$ exists and $\lim_{x\to c}[f\cdot g](x) = \lim_{x\to c} f(x)\cdot\lim_{x\to c} g(x)$. 5. $\lim_{x\to c}\left[\frac{f}{g}\right](x)$ exists and $\lim_{x\to c}\left[\frac{f}{g}\right](x) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$.
Limits for f ∘ g (Theorem 3.2.12)	Let I,J ⊆ ℝ be open intervals, let c ∈ I, let d ∈ J and let g: I − {c} → J − {d} and f: J − {d} → ℝ be functions. Suppose that $\lim_{y\to c} g(y) = d$ and that $\lim_{x\to d} f(x)$ exist. Then $\lim_{y\to c}(f\circ g)(y)$ exists, and $\lim_{y\to c}(f\circ g)(y) = \lim_{x\to d} f(x)$.
Limits: f ≤ g (Theorem 3.2.13)	Let I ⊆ ℝ be an open interval, let c ∈ I and let f,g: I − {c} → ℝ be functions. Suppose that f(x) ≤ g(x) for all x ∈ I − {c}. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$.
Squeeze Theorem for Functions (Theorem 3.2.14)	Let I ⊆ ℝ be an open interval, let c ∈ I and let f,g,h: I − {c} → ℝ be functions. Suppose that f(x) ≤ g(x) ≤ h(x) for all x ∈ I − {c}. If $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$ for some L ∈ ℝ, then $\lim_{x\to c} g(x)$ exists and $\lim_{x\to c} g(x) = L$.





Left/Right Hand Limits (Definition 3.2.15)	Let I ⊆ ℝ be an interval, let c ∈ I, let f: I − {c} → ℝ be a function and let L ∈ ℝ. 1. Suppose that c is not a right endpoint of I. The number L is the **right-hand limit** of f at c, written $$\lim_{x \to c+} f(x) = L,$$ if for each ε > 0, there is some δ > 0 such that x ∈ I − {c} and c < x < c + δ imply |f(x) − L| < ε. If $\lim_{x \to c+} f(x) = L$, we also say that f converges to L as x goes to c from the right. If f converges to some real number as x goes to c from the right, we say that $\lim_{x \to c+} f(x)$ exists. 2. Suppose that c is not a left endpoint of I. The number L is the **left-hand limit** of f at c, written $$\lim_{x \to c-} f(x) = L,$$ if for each ε > 0, there is some δ > 0 such that x ∈ I − {c} and c − δ < x < c imply |f(x) − L| < ε. If $\lim_{x \to c-} f(x) = L$, we also say that f converges to L as x goes to c from the left. If f converges to some real number as x goes to c from the left, we say that $\lim_{x \to c-} f(x) = L$, exists. 3. A **one-sided limit** is either a right-hand limit or a left-hand limit.
All 3 Limits are Equal (Lemma 3.2.17)	Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I − {c} → ℝ be a function. Then $\lim_{x \to c} f(x)$ exists if and only if $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exist and are equal, and if these three limits exist then they are equal.
y = mx + b (Exercise 3.2.1)	Let m,b,c ∈ ℝ. Using only the definition of limits, prove that $$\lim_{x \to c}(mx + b) = mc + b$$
Exercise 3.2.5	Let J ⊆ I ⊆ ℝ be open intervals, let c ∈ J and let f: I − {c} → ℝ be a function. Prove that $\lim_{x \to c} f(x)$ exists if and only if $\lim_{x \to c} f\|_J(x)$ exists, and if these limits exist, then they are equal.

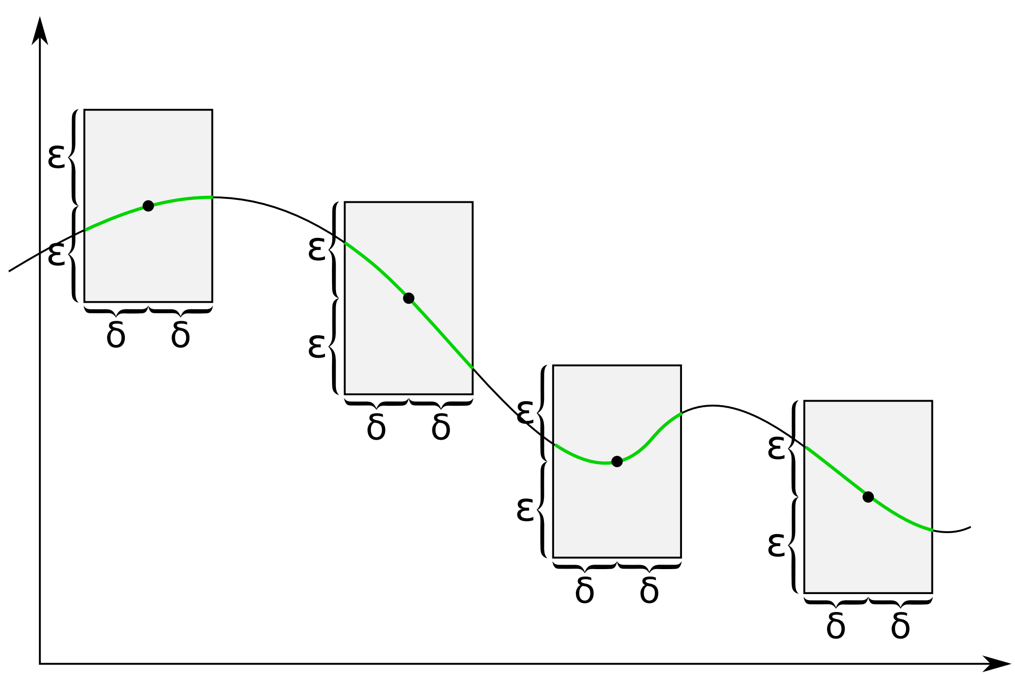
Ch. 3.3 Continuity

Theorem / Lemma / Corollary / Definition / Examples	Description
Continuity: ε, δ (Definition 3.3.1)	Let $A \subseteq \mathbb{R}$ be a set, and let $f: A \to \mathbb{R}$ be a function. 1. Let $c \in A$. The function f is **continuous** at c if for each $\varepsilon > 0$, there is some $\delta > 0$ such that $x \in A$ and $\|x - c\| < \delta$ imply $\|f(x) - f(c)\| < \varepsilon$. The function f is **discontinuous** at c if f is not continuous at c; in that case we also say that f has a discontinuity at c. 2. The function f is continuous if it is continuous **at every number** in A. The function f is discontinuous if it is not continuous.
Continuity: f(c) (Lemma 3.3.2)	Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$ and let $f: I \to \mathbb{R}$ be a function. Then f is continuous at c if and only if $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} f(x) = f(c)$.
Logical Form of Continuity	$(\forall c \in A)$[f is continuous at c] which can be written completely in symbols as $(\forall c \in A)\ (\forall \varepsilon > 0)\ (\exists \delta > 0)\ [(x \in A \wedge \|x - c\| < \delta) \to \|f(x) - f(c)\| < \varepsilon]$. The order of the quantifiers is crucial. Applies where we can find δ that depends upon ε and c.
Example 3.3.3	(1) $f(x) = mx + b$ (2) $p(x) = 1/x$ (3) Standard elementary functions (that is, polynomials, power functions, logarithms, exponentials and trigonometric functions). All of these functions are continuous. (4) $y = \tan(x)$ (5) $g(x) = \|x\|/x$ (6) $r(x) = 1$ or 0 (7) $s(x) = 1/q$
Sign-Preserving Property for Continuous Functions (Theorem 3.3.4)	Let $A \subseteq \mathbb{R}$ be a non-empty set, let $c \in A$ and let $f: A \to \mathbb{R}$ be a function. Suppose that f is continuous at c. 1. If $f(c) > 0$, then there is some $M > 0$ and some $\delta > 0$ such that $x \in A$ and $\|x - c\| < \delta$ imply $f(x) > M$. 2. If $f(c) < 0$, then there is some $N < 0$ and some $\delta > 0$ such that $x \in A$ and $\|x - c\| < \delta$ imply $f(x) < N$.
+, -, ·, ÷ Continuous at x = c (Theorem 3.3.5)	Let $A \subseteq \mathbb{R}$ be a non-empty set, let $c \in A$, let $f,g: A \to \mathbb{R}$ be functions and let $k \in \mathbb{R}$. Suppose that f and g are continuous at c. 1. $f + g$ is continuous at c. 2. $f - g$ is continuous at c. 3. $k \cdot f$ is continuous at c. 4. $f \cdot g$ is continuous at c. 5. If $g(c) \neq 0$, then f/g is continuous at c.

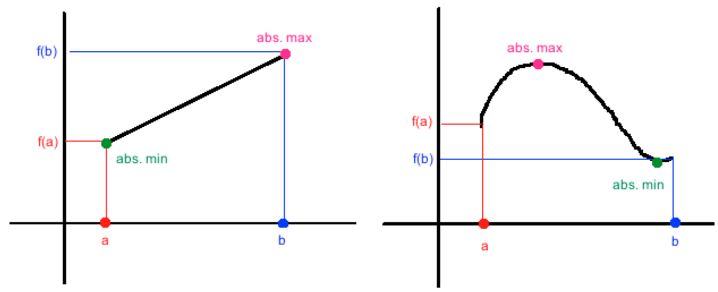
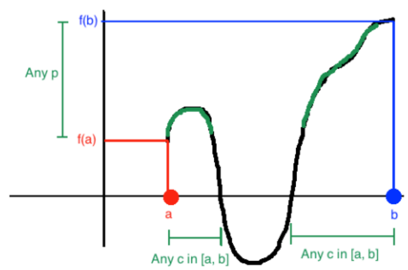
+, -, ·, ÷ Continuous Everywhere (Corollary 3.3.6)	Let A ⊆ ℝ be a non-empty set, let f,g: A → ℝ be functions and let k ∈ ℝ. Suppose that f and g are continuous. Then f + g, f − g, k · f and f · g are continuous, and if g(x) ≠ 0 for all x ∈ I then f / g is continuous.
Example 3.3.7	(1) $f_n(x) = x^n$ (2) p(x) = 1/x
Composite Functions (f ◦ g) (Theorem 3.3.8)	Let A,B ⊆ ℝ be non-empty sets, let c ∈ A and let g: A → B and f: B → ℝ be functions. 1. Suppose that A is an open interval. If $\lim_{x\to c} g(x)$ exists and is in B, and if f is continuous at $\lim_{x\to c} g(x)$, then $\lim_{x\to c} f\big(g(c)\big)$ = $f\Big(\lim_{x\to c} g(x)\Big)$. 2. If g is continuous at c, and if f is continuous at g(c), then f ◦ g is continuous at c. 3. If g and f are continuous, then f ◦ g is continuous.
Composition of Two Discontinuous Functions (Example 3.3.9)	(1) h(x) = 1 or 0, k(x) = 2 or 0 m → Better = Continuous (2) r(x) = 1 or 0, s(x) = 1/q → Worse Discontinuity
Pasting Lemma (Lemma 3.3.10)	Let [a,b] ⊆ ℝ and [b, c] ⊆ ℝ be non-degenerate closed bounded intervals, and let f: [a,b] → ℝ and g: [b,c] → ℝ be functions. Let h: [a,c] → ℝ be defined by h(x) = (f(x), if x ∈ [a,b], g(x), if x ∈ [b,c]. If f and g are continuous, and if f(b) = g(b), then h is continuous.
Extension of a Function (Example 3.3.11)	f(x) = x → Can be extended p(x) = 1/x → Cannot be extended

Ch. 3.4 Uniform Continuity

Lemma / Corollary / Definition / Examples	Description
Uniformly Continuous (UC) (Definition 3.4.1)	Let $A \subseteq \mathbb{R}$ be a set, and let $f: A \to \mathbb{R}$ be a function. The function f is **uniformly continuous** if for each $\varepsilon > 0$, there is some $\delta > 0$ such that $x, y \in A$ and $\lvert x - y\rvert < \delta$ imply $\lvert f(x) - f(y)\rvert < \varepsilon$.
Logical Form of UC	$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x \in A)\ (\forall y \in A)\ [\lvert x - y\rvert < \delta \to \lvert f(x) - f(y)\rvert < \varepsilon]$ The order of the quantifiers is crucial. Applies where we can find δ that depends only upon ε, and not c.
UC → C (Lemma 3.4.2)	Let $A \subseteq \mathbb{R}$ be a set, and let $f: A \to \mathbb{R}$ be a function. If f is uniformly continuous, then f is continuous.
Example 3.4.3	(1) $f(x) = mx + b$ → Is UC (2) $g(x) = 1/x$ where $x \in \mathbb{R} - \{0\}$ → Is not UC (3) $g(x) = 1/x$ where $x \in (1, \infty)$ → Is UC
Close Bounded Interval C → UC (Theorem 3.4.4)	Let $C \subseteq \mathbb{R}$ be a closed bounded interval, and let $f: C \to \mathbb{R}$ be a function. If f is continuous, then f is uniformly continuous.
UC → Bounded (Theorem 3.4.5)	Let $A \subseteq \mathbb{R}$ be a non-empty set, and let $f: A \to \mathbb{R}$ be a function. Suppose that A is bounded. If f is uniformly continuous, then f is bounded.
C → Bounded (Corollary 3.4.6)	Let $C \subseteq \mathbb{R}$ be a closed bounded interval, and let $f: C \to \mathbb{R}$ be a function. If f is continuous, then f is bounded.



Ch. 3.5 Two Important Theorems

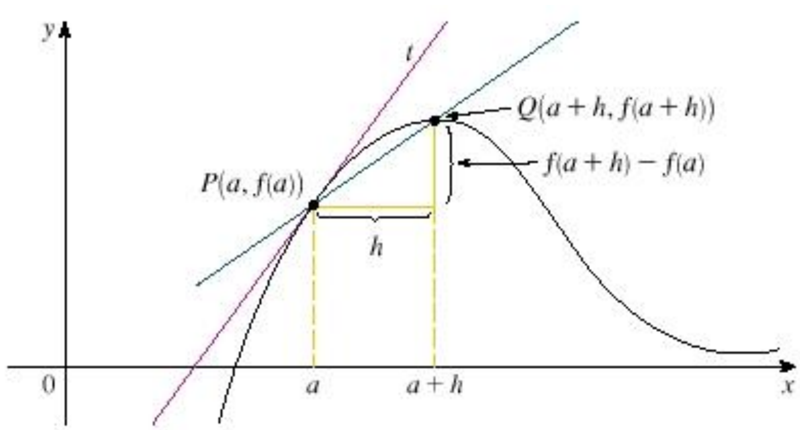
Axiom / Theorem / Lemma / Definition	Description
Extreme Value Theorem: Min. and Max. Exist (Theorem 3.5.1)	Let C ⊆ ℝ be a closed bounded interval, and let f: C → ℝ be a function. Suppose that f is continuous. Then there are x_{min}, x_{max} ∈ C such that $f(x_{min}) \le f(x) \le f(x_{max})$ for all x ∈ C. 
Intermediate Value Theorem (Theorem 3.5.2)	Let [a,b] ⊆ ℝ be a closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is continuous. Let r ∈ ℝ. If r is strictly between f(a) and f(b), then there is some c ∈ (a,b) such that f(c) = r. We can assume f(a) < r < f(b). 
Contrapositive for a Proof (Lemma 3.5.3)	Let F be an ordered field. **Suppose that F does not satisfy the Least Upper Bound Property**. Let A ⊆ F be a non-empty set such that A is bounded above, but A has no least upper bound. Let a ∈ A, and let b ∈ F be an upper bound of A. Let Q = {x ∈ [a,b] | x is an upper bound of A} and P = [a,b] − Q. 1. P ∪ Q = [a,b] and P ∩ Q = ∅. 2. a < b, and A ∩ [a,b] ⊆ P, and a ∈ P, and b ∈ Q. 3. If x ∈ P and z ∈ Q, then x < z. 4. If x ∈ P, then there is some y ∈ P such that x < y. If z ∈ Q, then there is some w ∈ Q such that w < z. 5. The set P does not have a least upper bound, and the set Q does not have a greatest lower bound.
Theorem 3.5.4	The following are equivalent. a1. The Least Upper Bound Property. a2. The Greatest Lower Bound Property. b. The Heine–Borel Theorem. c. The Extreme Value Theorem. d. The Intermediate Value Theorem.

Ch. 4.2 The Derivative

Definition / Theorem / Example	Description
Definition of Derivative with x - c (Definition 4.2.1)	Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I → ℝ be a function. 1. The function f is **differentiable** at c if $$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$ exists; if this limit exists, it is called the **derivative** of f at c, and it is denoted f'(c). 2. The function f is **differentiable** if it is differentiable at every number in I. If f is differentiable, the **derivative** of f is the function f': I → ℝ whose value at x is f'(x) for all x ∈ I.
Definition of Derivative with h (Lemma 4.2.2)	Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I → ℝ be a function. Then f is differentiable at c if and only if $$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$ exists, and if this limit exists it equals f'(c).
Example 4.2.3	(1) f(x) = mx + b so (mx + b)' = m. (2) g(x) = x^2 so g'(x) = 2x (3) k(x) = |x| so k'(x) does not exist unless x ∈ (0, ∞).
Differentiable → Continuous (Theorem 4.2.4)	Let I ⊆ ℝ be an open interval, and let f: I → ℝ be a function. Let c ∈ I. If f is differentiable at c, then f is continuous at c. If f is differentiable, then f is continuous.
Continuous vs. Differentiable (Example 4.2.5)	(1) f(x) = {x^2 sin (1/x^2), if x ≠ 0} {0, if x = 0}. So, f' exists everywhere, but f' is not continuous. (2) g(x) = { x^2, if x ≥ 0} {- x^2, if x < 0} So, g' is continuous, however g' is not differentiable.

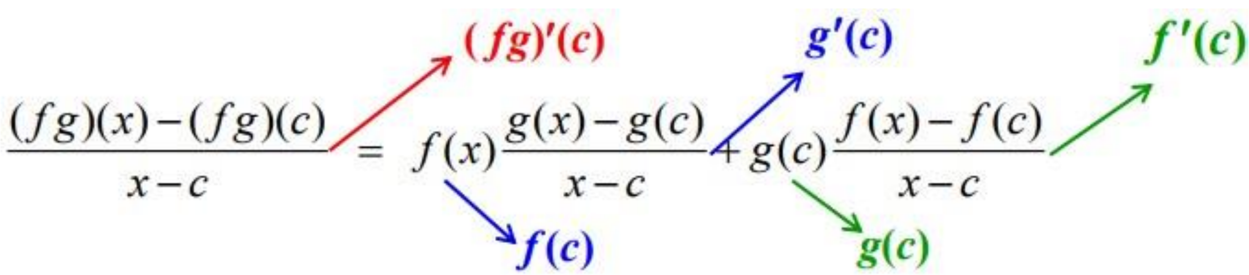
n^{th} Derivatives (Definition 4.2.6)	Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$ and let $f: I \to \mathbb{R}$ be a function. Suppose that f is differentiable at c. The function f is **twice differentiable** at c if f' is differentiable at c. If f' is differentiable at c, the derivative (f')'(c) is called the **second derivative** of f at c, and it is denoted f''(c). The function f is **twice differentiable** if it is twice differentiable at every number in I. If f is twice differentiable, the second derivative of f is the function $f'': I \to \mathbb{R}$ whose value at x is f''(x) for all $x \in I$. The n^{th} derivative of f for all $n \in \mathbb{N}$ is defined as follows, using Definition by Recursion. If f is differentiable at c, the **first derivative** of f at c is simply the derivative of f at c. Suppose that f is n−1 times differentiable at c. The (n−1)-st derivative of f at c is denoted $f^{(n-1)}(c)$. The function f is **n times differentiable** at c if $f^{(n-1)}$ is differentiable at c. If $f^{(n-1)}$ is differentiable at c, the derivative $(f^{(n-1)})'(c)$ is called the **n^{th} derivative** of f at c, and it is denoted $f^{(n)}(c)$. The function f is n times differentiable if it is n times differentiable at every number in I. If f is n times differentiable, the n^{th} derivative of f is the function $f^{(n)}: I \to \mathbb{R}$ whose value at x is $f^{(n)}(x)$ for all $x \in I$. The **0^{th} derivative** of f is $f^{(0)} = f$.
Continuously/Infinitely Differentiable (Definition 4.2.7)	Let $I \subseteq \mathbb{R}$ be an open interval, and let $f: I \to \mathbb{R}$ be a function. The function f is **continuously differentiable** if f is differentiable and f' is continuous. Let $n \in \mathbb{N}$. The function f is **continuously differentiable of order** n if $f^{(i)}$ exists and is continuous for all $i \in \{1, ..., n\}$. The function f is **infinitely differentiable** (also called **smooth**) if $f^{(i)}$ exists all $i \in \mathbb{N}$.

One-Sided Derivatives (Definition 4.2.8)	Let I ⊆ ℝ be a non-degenerate interval, let c ∈ I and let f: I → ℝ be a function. 1. Suppose that c is a left endpoint of I. The function f is **differentiable** at c if the limit $$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$ exists; if this limit exists, it is called the **one-sided derivative** of f at c, and it is denoted f'(c). 2. Suppose that c is a right endpoint of I. The function f is **differentiable** at c if the limit $$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$ exists; if this limit exists, it is called the **one-sided derivative** of f at c, and it is denoted f'(c). 3. The function f is **differentiable** if the restriction of f to the interior of I is differentiable in the usual sense, and if f is differentiable at the endpoints of I in the sense of Parts (1) and (2) of this definition if there are endpoints.
Symmetric Derivative (Exercise 4.2.7)	Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I → ℝ be a function. The function f is **symmetrically differentiable** at c if $$\lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h}$$ exists; if this limit exists, it is called the **symmetric derivative** of f at c.

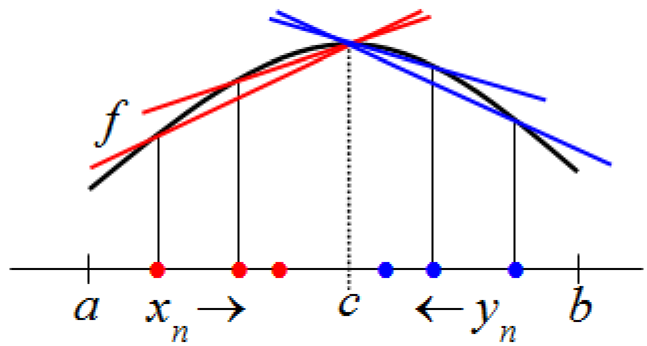
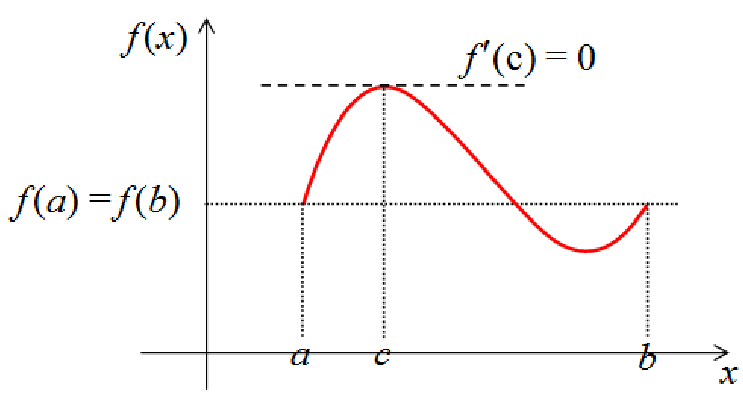


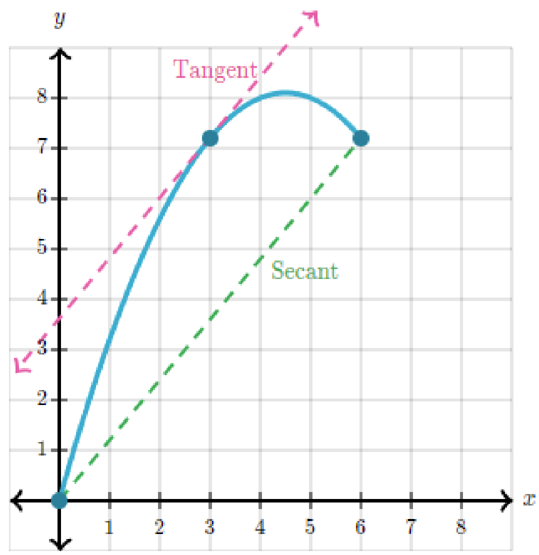
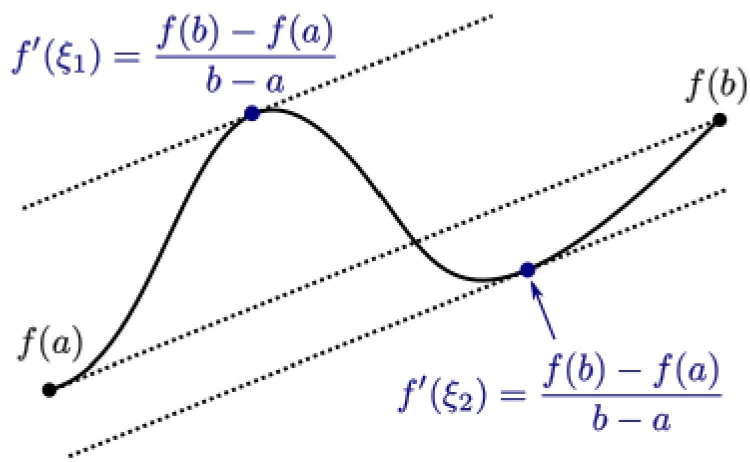
Ch. 4.3 Computing Derivatives

Theorem / Corollary	Description
Derivatives: +, −, •, ÷ (Theorem 4.3.1)	Let I ⊆ ℝ be an open interval, let c ∈ I, let f,g: I → ℝ be functions and let k ∈ ℝ. Suppose that f and g are differentiable at c. 1. **f + g** is differentiable at c and [f + g]’(c) = f’(c) + g’(c). 2. **f − g** is differentiable at c and [f − g]’(c) = f’(c) − g’(c). 3. **kf** is differentiable at c and [kf]’(c) = k f’(c). 4. (**Product Rule**) fg is differentiable at c and [fg]’(c) = f’(c)g(c) + f(c)g’(c). 5. (**Quotient Rule**) If g(c) ≠ 0, then f/g is differentiable at c and $\left[\frac{f}{g}\right]'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$
Entire Function (Corollary 4.3.2)	Let I ⊆ ℝ be an open interval, let f,g: I → ℝ be functions and let k ∈ ℝ. If f and g are differentiable, then f + g, f − g, kf and fg are differentiable, and if g(x) ≠ 0 for all x ∈ I then f/g is differentiable.
Chain Rule (Theorem 4.3.3)	Let I,J ⊆ ℝ be open intervals, let c ∈ I and let f: I → J and g: J → ℝ be functions. Suppose that f is differentiable at c, and that g is differentiable at f(c). Then g ∘ f is differentiable at c and [g ∘ f]’(c) = g’ (f(c))· f’(c).
Chain Rule Differentiable (Corollary 4.3.4)	Let I,J ⊆ ℝ be open intervals, and let f: I → J and g: J → ℝ be functions. If f and g are differentiable, then g ∘ f is differentiable.



Ch. 4.4 The Mean Value Theorem

Axiom / Theorem / Lemma / Definition	Description
Min/Max at a Point, f'(c) = 0 (Lemma 4.4.1)	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let c ∈ (a,b) and let f: [a,b] → ℝ be a function. Suppose that f is differentiable at c. If either f(c) ≥ f(x) for all x ∈ [a,b] or f(c) ≤ f(x) for all x ∈ [a,b], then f'(c) = 0. 
f'(c) = 0, But Not a Min/Max (Example 4.4.2)	Let f: [−1,1] → ℝ be defined by f(x) = x^3 for all x ∈ [−1,1]. It can be verified using the definition of derivatives that f'(0) = 0; the details are left to the reader. On the other hand, it is certainly not the case that f(0) ≥ f(x) for all x ∈ [−1,1], or that f(0) ≤ f(x) for all x ∈ [−1,1].
Rolle's Theorem: f(a) = f(b) (Lemma 4.4.3)	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is continuous on [a,b] and differentiable on (a,b). If f(a) = f(b), then there is some c ∈ (a,b) such that f'(c) = 0.  **Note**: Rolle's Theorem is a special case of the Mean Value Theorem where f(a) = f(b).

Mean Value Theorem (Average Slope) (Theorem 4.4.4)	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is <u>continuous</u> on [a,b] and <u>differentiable</u> on (a,b). Then there is some c ∈ (a,b) such that $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$   **Note**: The Mean Value Theorem is a special case of Cauchy's Mean Value Theorem where g(x) = x. **Note**: The Mean Value Theorem is a special case of Taylor's Theorem where n = 0, c = a, and x = b

Cauchy's Mean Value Theorem (Theorem 4.4.5)	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f,g: [a,b] → ℝ be functions. Suppose that f and g are continuous on [a,b] and differentiable on (a,b). Then there is some c ∈ (a,b) such that $$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$ 
Cauchy → Mean Value Theorem	The Mean Value Theorem is the special case of Cauchy's Mean Value Theorem (Theorem 4.4.5) where the function g is defined by g(x) = x for all x ∈ [a,b].
Taylor's Theorem (Theorem 4.4.6)	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let c ∈ (a,b), let f: [a,b] → ℝ be a function and let n ∈ ℕ ∪ {0}. Suppose that $f^{(k)}$ exists and is continuous on [a,b] for each k ∈ {0, ..., n}, and that $f^{(n+1)}$ exists on (a,b). Let x ∈ [a,b]. Then there is some p strictly between x and c (except that p = c when x = c) such that $$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(k+1)}(p)}{(n+1)!}(x-c)^{n+1}.$$.
Parallel Functions (Lemma 4.4.7)	Let I ⊆ ℝ be a non-degenerate interval, and let f,g: I → ℝ be function. Suppose that f and g are continuous on I and differentiable on the interior of I. 1. f'(x) = 0 for all x in the interior of I if and only if f is constant on I. 2. f'(x) = g'(x) for all x in the interior of I if and only if there is some C ∈ ℝ such that f(x) = g(x) + C for all x ∈ I.
Antiderivative (F' = f) (Definition 4.4.8)	Let I ⊆ ℝ be an open interval, and let f: I → ℝ be a function. An **antiderivative** of f is a function F: I → ℝ such that F is differentiable and F' = f.

Constant of Integration (+ C) **(Corollary 4.4.9)**	Let I ⊆ ℝ be a non-degenerate open interval, and let f: I → ℝ be a function. If F,G: I → ℝ are antiderivatives of f, then there is some C ∈ ℝ such that F(x) = G(x) + C for all x ∈ I.
Intermediate Value Theorem for Derivatives **(Theorem 4.4.10)**	Let I ⊆ ℝ be an open interval, and let f: I → ℝ be a function. Suppose that f is differentiable. Let a,b ∈ I, and suppose that a < b. Let r ∈ ℝ. If r is strictly between f'(a) and f'(b), then there is some c ∈ (a,b) such that f'(c) = r.
g(x) ≠ f'(x) **(Example 4.4.11)**	Let g: ℝ → ℝ be defined by $$g(x) = \begin{cases} 1, & if\ x \leq 1 \\ 2, & if\ x > 1. \end{cases}$$ Then g is not the derivative of any function, because it does not satisfy the conclusion of the Intermediate Value Theorem for Derivatives (Theorem 4.4.10).

Ch. 4.5 Increasing and Decreasing Functions, Part I: Local and Global Extrema

Axiom / Theorem / Lemma / Definition	Description
f(x) vs. Increasing / Decreasing / Monotone (Definition 4.5.1)	Let $A \subseteq \mathbb{R}$ be a set, and let $f: A \to \mathbb{R}$ be a function. 1. The function f is **increasing** if $x < y$ implies $f(x) \le f(y)$ for all $x,y \in A$. 2. The function f is **strictly increasing** if $x < y$ implies $f(x) < f(y)$ for all $x, y \in A$. 3. The function f is **decreasing** if $x < y$ implies $f(x) \ge f(y)$ for all $x,y \in A$. 4. The function f is **strictly decreasing** if $x < y$ implies $f(x) > f(y)$ for all $x, y \in A$. 5. The function f is **monotone** if it is either increasing or decreasing. 6. The function f is **strictly monotone** if it is either strictly increasing or strictly decreasing.
f'(x) vs. Increasing (Theorem 4.5.2)	Let $I \subseteq \mathbb{R}$ be a non-degenerate interval, and let $f: I \to \mathbb{R}$ be a function. Suppose that f is continuous on I and differentiable on the interior of I. 1. $f'(x) \ge 0$ for all x in the interior of I if and only if f is **increasing** on I. 2. If $f'(x) > 0$ for all x in the interior of I, then f is **strictly increasing** on I. 3. $f'(x) \le 0$ for all x in the interior of I if and only if f is **decreasing** on I. 4. If $f'(x) < 0$ for all x in the interior of I, then f is **strictly decreasing** on I.
Example 4.5.3	Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$ for all $x \in \mathbb{R}$. The function f is strictly increasing, as seen by Exercise 2.3.3 (1); that exercise does not make use of derivatives. However, we know by Exercise 4.3.5 that $f'(x) = 3x^2$ for all $x \in \mathbb{R}$, and hence $f'(0) = 0$. Therefore Theorem 4.5.2 (2) cannot be made into an "if and only if" statement. A similar example shows that Theorem 4.5.2 (4) cannot be made into an "if and only if" statement.

Local/Global Extremum (Definition 4.5.4)	Let $A \subseteq \mathbb{R}$ be a set, let $c \in A$ and let $f: A \to \mathbb{R}$ be a function. 1. The number c is a **local maximum** of f if there is some $\delta > 0$ such that $x \in A$ and $\lvert x - c \rvert < \delta$ imply $f(x) \leq f(c)$. 2. The number c is a **local minimum** of f if there is some $\delta > 0$ such that $x \in A$ and $\lvert x - c \rvert < \delta$ imply $f(x) \geq f(c)$. 3. The number c is a **local extremum** of f if it is either a local maximum or a local minimum. 4. The number c is a **global maximum** of f if $f(x) \leq f(c)$ for all $x \in A$. 5. The number c is a **global minimum** of f if $f(x) \geq f(c)$ for all $x \in A$. 6. The number c is a **global extremum** of f if it is either a global maximum or a global minimum.
Local Min/Max (Lemma 4.5.5)	Let $A \subseteq \mathbb{R}$ be a set, let $c \in A$ and let $f: A \to \mathbb{R}$ be a function. 1. If there is some $\delta > 0$ such that $f\rvert_{A \cap (c-\delta, c]}$ is increasing and $f\rvert_{A \cap [c, c+\delta)}$ is decreasing, then c is a **local maximum** of f. 2. If there is some $\delta > 0$ such that $f\rvert_{A \cap (c-\delta, c]}$ is decreasing and $f\rvert_{A \cap [c, c+\delta)}$ is increasing, then c is a **local minimum** of f.
Critical Point (Definition 4.5.6)	Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$ and let $f: I \to \mathbb{R}$ be a function. The number c is a **critical point** of f if either f is differentiable at c and f 0 (c) = 0, or f is not differentiable at c.
Extremum → Critical Point (Lemma 4.5.7)	Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$ and let $f: I \to \mathbb{R}$ be a function. If c is a **local extremum** of f, then c is a **critical point** of f.
Example 4.5.8	Let $f: [-1,1] \to \mathbb{R}$ be defined by $f(x) = x^3$ for all $x \in [-1,1]$. Because $f'(x) = 3x^2$ for all $x \in \mathbb{R}$, then $f'(0) = 0$, and hence 0 is a critical point of f. However, as remarked in Example 4.5.3, the function f is strictly increasing, and therefore 0 is neither a local maximum nor a local minimum of f.
First Derivative Test (Theorem 4.5.9)	Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$ and let $f: I \to \mathbb{R}$ be a function. Suppose that c is a critical point of f, and that f is continuous on I and differentiable on $I - \{c\}$. 1. Suppose that there is some $\delta > 0$ such that $x \in I$ and $c - \delta < x < c$ imply $f'(x) \geq 0$, and that $x \in I$ and $c < x < c + \delta$ imply $f'(x) \leq 0$. Then c is a **local maximum** of f. 2. Suppose that there is some $\delta > 0$ such that $x \in I$ and $c - \delta < x < c$ imply $f'(x) \leq 0$, and that $x \in I$ and $c < x < c + \delta$ imply $f'(x) \geq 0$. Then c is a **local minimum** of f. 3. Suppose that there is some $\delta > 0$ such that $x \in I - \{c\}$ and $\lvert x - c \rvert < \delta$ imply $f'(x) > 0$, or that $x \in I - \{c\}$ and $\lvert x - c \rvert < \delta$ imply $f'(x) < 0$. Then c is not a **local extremum** of f.

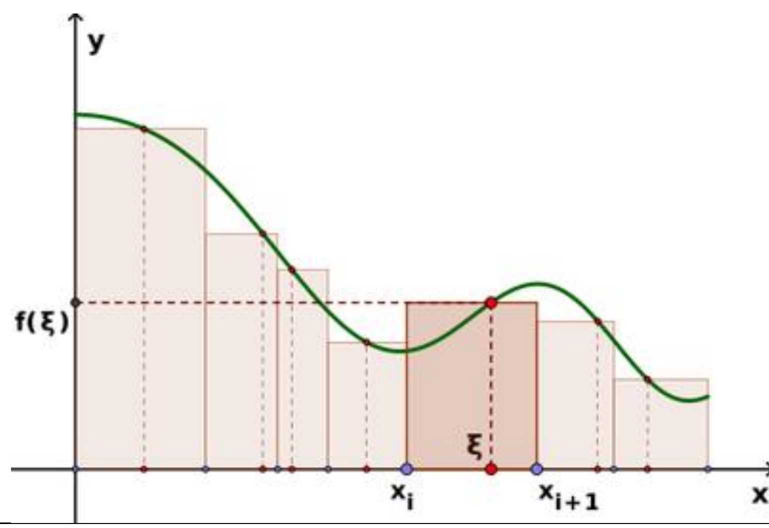
Second Derivative Test (Theorem 4.5.10)	Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$ and let $f: I \to \mathbb{R}$ be a function. Suppose that f is differentiable, that $f'(c) = 0$ and that f is twice differentiable at c. 1. If $f''(c) > 0$, then c is a local minimum of f. 2. If $f'(c) < 0$, then c is a local maximum of f.
Example 4.5.11	(1) Let $f,g: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$ and $g(x) = x^4$ for all $x \in \mathbb{R}$. It is straightforward to verify that $f'(0) = 0$ and $g'(0) = 0$, and that $f''(0) = 0$ and $g'(0) = 0$. Because $x^4 = (x^2)^2 \geq 0$ for all $x \in \mathbb{R}$, then 0 is a local (and also global) minimum of g. As noted in Example 4.5.8, the number 0 is not a local extremum of f.
	(2) Let $k: \mathbb{R} \to \mathbb{R}$ be defined by $k(x) = \lvert x \rvert$ for all $x \in \mathbb{R}$. We saw in Example 4.2.3 (3) that k is not differentiable at 0, and hence 0 is a critical point of k. We also saw that $k'(x) = -1$ for all $x \in (-\infty,0)$, and $k'(x) = 1$ for all $x \in (0, \infty)$. Because k is not differentiable at 0, we cannot apply the Second Derivative Test (Theorem 4.5.10) to k at 0. However, the First Derivative Test (Theorem 4.5.9) can still be applied, and we see that 0 is a local minimum of k, which is just what we would expect by looking at the graph of k.
Local → Global (Theorem 4.5.12)	Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$ and let $f: I \to \mathbb{R}$ be a function. Suppose that f is continuous, and that c is the **only critical point** of f. 1. If c is a local maximum, then it is a global maximum. 2. If c is a local minimum, then it is a global minimum.

Ch. 4.6 Increasing and Decreasing Functions, Part II: Further Topics

Axiom / Theorem / Lemma / Definition	Description
Not Differentiable (Example 4.6.1)	Let f: $\mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$ for all $x \in \mathbb{R}$. Intuitively, we know that the function f is bijective, and hence it has an inverse function $f^{-1}: \mathbb{R} \to \mathbb{R}$, which we write as $f^{-1}(x) = \sqrt[3]{x}$ for all $x \in \mathbb{R}$. Moreover, we know that the graph of f^{-1} is obtained from the graph of f by reflection in the line y = x. Because f has a horizontal tangent line at the origin, then the graph of f^{-1} has a vertical tangent line at x = 0, which makes it not differentiable at x = 0.
Bounded Intervals (Lemma 4.6.2)	Let $I \subseteq \mathbb{R}$ be a non-degenerate open interval, and let $f: I \to \mathbb{R}$ be a function. Suppose that f is **strictly monotone**. 1. The function $f: I \to f(I)$ is **bijective**. 2. Suppose that f is continuous. Then f(I) is a non-degenerate open interval, and one of the following holds: a. If the interval f(I) is bounded, then f(I) = (glb f(I),lub f(I)). b. If the interval f(I) is bounded above but is not bounded below, then $f(I) = (-\infty, \text{lub } f(I))$. c. If the interval f(I) is bounded below but is not bounded above, then $f(I) = (\text{glb } f(I), \infty)$. d. If the interval f(I) is not bounded above and is not bounded below, then $f(I) = \mathbb{R}$.
Example 4.6.3	We want to show that the square root function is continuous. Let f: $(0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^2$ for all $x \in \mathbb{R}$. By Exercise 3.5.6 (1) we see that f is strictly increasing, and by Example 3.3.7 (1) we see that f is continuous. Exercise 3.5.6 implies that $f((0, \infty)) = (0, \infty)$. It then follows from Lemma 4.6.2 (3) that $f^{-1}: (0, \infty) \to (0, \infty)$ is continuous and strictly increasing. By Definition 2.6.10 we see that $f^{-1}(x) = \sqrt[2]{x}$ for all $x \in (0, \infty)$. The continuity of this function could also be shown directly by an ε–δ proof, but Lemma 4.6.2 allows us to avoid that.
Inverse Derivatives (Theorem 4.6.4)	Let $I \subseteq \mathbb{R}$ be a non-degenerate open interval, and let $f: I \to \mathbb{R}$ be a function. Suppose that f is differentiable, and that $f'(x) \neq 0$ for all $x \in I$. 1. The function f is strictly monotone. 2. The function $f: I \to f(I)$ is bijective. 3. The function $f^{-1}: f(I) \to I$ is differentiable. 4. The derivative of f^{-1} is given by $$[f^{-1}]'(x) = \frac{1}{f'(f^{-1}(x))}$$ for all $x \in f(I)$.

Secant Line (Definition 4.6.5)	Let $I \subseteq \mathbb{R}$ be an open interval, let $a,b \in I$ and let $f: I \to \mathbb{R}$ be a function. Suppose that $a < b$. The **secant line** through $(a, f(a))$ and $(b, f(b))$ is the function $S_{a,b}: \mathbb{R} \to \mathbb{R}$ defined by $$S_{a,b}(x) = f(a)\frac{b-x}{b-a} + f(b)\frac{x-a}{b-a}$$ for all $x \in \mathbb{R}$. The slope of the secant line through $(a, f(a))$ and $(b, f(b))$, denoted $M_{a,b}$, is defined by $$M_{a,b} = \frac{f(b)-f(a)}{b-a}.$$
Function vs Secant Line (Theorem 4.6.6)	Let $I \subseteq \mathbb{R}$ be an open interval, and let $f: I \to \mathbb{R}$ be a function. The following are equivalent. a. If $a,b \in I$ and $a < b$, then $f(x) \leq S_{a,b}(x)$ for all $x \in [a,b]$ (Function Lies Below Its Secant Lines). b. If $a,b,c \in I$ and $a < b < c$, then $M_{a,b} \leq M_{b,c}$ (Function Has Increasing Secant Line Slopes).
Concave Up (Definition 4.6.7)	Let $I \subseteq \mathbb{R}$ be an open interval, and let $f: I \to \mathbb{R}$ be a function. The function f is **concave up** if either of the two conditions in Theorem 4.6.6 hold.
Theorem 4.6.8	Let $I \subseteq \mathbb{R}$ be an open interval, and let $f: I \to \mathbb{R}$ be a function. 1. Suppose that f is differentiable. Then the two conditions in Theorem 4.6.6 hold if and only if f' is increasing on I. 2. Suppose that f is twice differentiable. Then the two conditions in Theorem 4.6.6 hold if and only if $f''(x) \geq 0$ for all $x \in I$.

Ch. 5.2 The Riemann Integral

Axiom / Theorem / Lemma / Definition	Description
Definition 5.2.1	Let $[a,b] \subseteq \mathbb{R}$ be a non-degenerate closed bounded interval. 1. A **partition** of $[a,b]$ is a set $P = \{x_0, x_1, ..., x_n\}$ such that $a = x_0 < x_1 < \cdots < x_n = b$, for some $n \in \mathbb{N}$. 2. If $P = \{x_0, x_1, ..., x_n\}$ is a partition of $[a,b]$, the **norm** (also called the **mesh**) of P, denoted $\|P\|$, is defined by $\|P\| = \max\{x_1 - x_0, x_2 - x_1, ..., x_n - x_{n-1}\}$. 3. If $P = \{x_0, x_1, ..., x_n\}$ is a partition of $[a,b]$, a **representative set** of P is a set $T = \{t_1, t_2, ..., t_n\}$ such that $t_i \in [x_{i-1}, x_i]$ for all $i \in \{1, ..., n\}$.
Riemann Sum **(Definition 5.2.2)**	Let $[a,b] \subseteq \mathbb{R}$ be a non-degenerate closed bounded interval, let $f: [a,b] \to \mathbb{R}$ be a function, let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$ and let $T = \{t_1, t_2, ..., t_n\}$ be a representative set of P. The **Riemann sum** of f with respect to P and T, denoted S(f,P,T), is defined by $$S(f,P,T) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$ 
Example 5.2.3	(1) $f(x) = x^2$
	(2) r(x) = {1 or 0}
Definition of Integrability **(ε- δ)** **(Definition 5.2.4)**	Let $[a,b] \subseteq \mathbb{R}$ be a non-degenerate closed bounded interval, let $f: [a,b] \to \mathbb{R}$ be a function and let $K \in \mathbb{R}$. The number K is the **Riemann integral** of f, written $$\int_a^b f(x)\,dx = K,$$ if for each $\varepsilon > 0$, there is some $\delta > 0$ such that if P is a partition of $[a,b]$ with $\|P\| < \delta$, and if T is a representative set of P, then $\|S(f,P,T) - K\| < \varepsilon$. If the Riemann integral of f exists, we say that f is **Riemann integrable**.

Unique K **(Lemma 5.2.5)**	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. If f is Riemann integrable, then there is a unique K ∈ ℝ such that $$\int_a^b f(x)\,dx = K.$$
Example 5.2.6	(1) f(x) = c
	(2) g(x) = {7 or 0}
	(3) r(x) = {0 or 1}
	(4) s(x) = {1/q or 0}
	(5) v(x) = {0 or 1}
Exercise 5.2.1	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let ε > 0. Prove that there is a partition R of [a,b] such that $\|R\| < \varepsilon$.

Ch. 5.3 Elementary Properties of the Reimann Integral

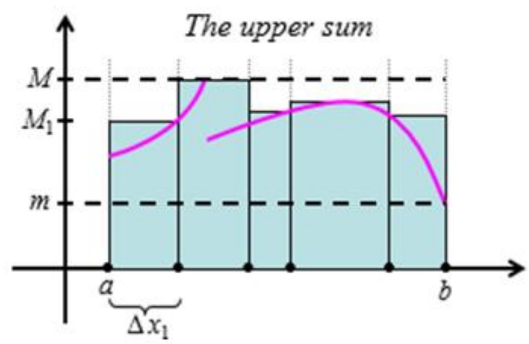
Axiom / Theorem / Lemma / Definition	Description
Integration: +, -, k (Theorem 5.3.1)	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let f,g: [a,b] → ℝ be functions and let k ∈ ℝ. Suppose that f and g are integrable. 1. f + g is integrable and $$\int_a^b [f+g](x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$ 2. f − g is integrable and $$\int_a^b [f-g](x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$ 3. k·f is integrable and $$\int_a^b [kf](x)\,dx = k\int_a^b f(x)\,dx.$$ 4. $\int_a^b k\,dx = k(b-a)$.
Theorem 5.3.2	Let [a,b] ⊆ ℝ be a closed bounded interval, and let f,g: [a,b] → ℝ be functions. Suppose that f and g are integrable. 1. If f(x) ≥ 0 for all x ∈ [a,b], then $\int_a^b f(x)\,dx \geq 0$. 2. If f(x) ≥ g(x) for all x ∈ [a,b], then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$. 3. Let m,M ∈ ℝ. If m ≤ f(x) for all x ∈ [a,b], then m(b−a) ≤ $\int_a^b f(x)\,dx$, and if f(x) ≤ M for all x ∈ [a,b], then $\int_a^b f(x)\,dx \leq M(b-a)$.
Integrable → Bounded (Theorem 5.3.3)	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. If f is **integrable**, then f is **bounded**.

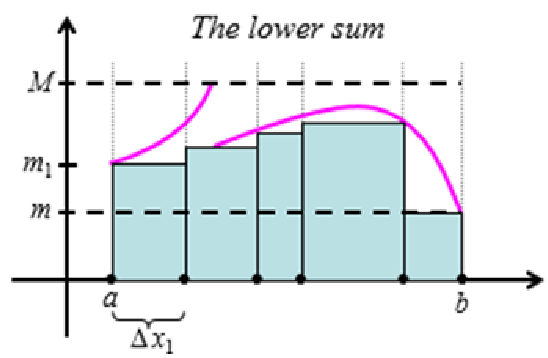
Ch. 5.4 Upper Sums and Lower Sums

Axiom / Theorem / Lemma / Definition	Description
Refinement (Definition 5.4.1)	Let $[a,b] \subseteq \mathbb{R}$ be a non-degenerate closed bounded interval, and let P and Q be partitions of [a,b]. The partition Q is a **refinement** of P if $P \subseteq Q$.
Example 5.4.2	The sets P = {0, ½ , 1}, and Q = {0, ¼, ½, ¾, 1} and $\mathbb{R}$ = {0, 1/3, 2/3 , 1} are partitions of [0,1]. Then Q is a refinement of P, but $\mathbb{R}$ is not a refinement of P.
Norm of a Refinement (Lemma 5.4.3)	Let $[a,b] \subseteq \mathbb{R}$ be a non-degenerate closed bounded interval, and let P and Q be partitions of [a,b]. 1. $P \cup Q$ is a partition of [a,b], and $P \cup Q$ is a refinement of each of P and Q. 2. If Q is a refinement of P, then $\|\|Q\|\| \leq \|\|P\|\|$.

<table>
<tr><td>Upper/Lower Sums (Definition 5.4.4)</td><td>
Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let f: [a,b] → ℝ be a function and let P = {x_0, x_1, ..., x_n} be a partition of [a,b]. Suppose that f is bounded.

1. For each i ∈ {1, ..., n}, let $M_i(f)$ = lub f([x_{i-1}, x_i]) and $m_i(f)$ = glb f([x_{i-1}, x_i]). If it is necessary to indicate the partition being used, we will write $M_i^P(f)$and $m_i^P(f)$.

2. The upper sum of f with respect to P, denoted U(f,P), is defined by
$$U(f,P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$
and the lower sum of f with respect to P, denoted
$$L(f,P) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1})$$
NOTE: An upper sum of a continuous function, f, takes a point c_i in each subinterval where the maximum value of f is achieved.
A lower sum takes the minimum value of f for each subinterval.




</td></tr>
<tr><td>Example 5.4.5</td><td>(1) f(x) = x^2</td></tr>
<tr><td></td><td>(2) g(x) = {7 or 0}</td></tr>
<tr><td></td><td>(3) r(x) = {1 or 0}</td></tr>
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<table>
<tr><td>Lemma 5.4.6</td><td>Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let f: [a,b] → ℝ be a function and let P be a partition of [a,b]. Suppose that f is bounded.
1. If T is a representative set of P, then L(f,P) ≤ S(f,P,T) ≤ U(f,P).
2. If ℝ is a refinement of P, then L(f,P) ≤ L(f,ℝ) ≤ U(f,ℝ) ≤ U(f,P).
3. If Q is a partition of [a,b], then L(f,P) ≤ U(f,Q).</td></tr>
<tr><td>Integrable Equivalents (Theorem 5.4.7)</td><td>Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is bounded. The following are equivalent.
a. The function f is integrable.
b. For each ε > 0, there is some δ > 0 such that if P is a partition of [a,b] with ||P|| < δ, then U(f,P) − L(f,P) < ε.
c. For each ε > 0, there is some partition P of [a,b] such that U(f,P) − L(f,P) < ε.</td></tr>
<tr><td>Upper/Lower Integral (Definition 5.4.8)</td><td>Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is bounded.
The upper integral of f, denoted $\overline{\int_a^b} f(x)dx$, is defined by
$$\overline{\int_a^b} f(x)dx = \mathrm{glb}\{\mathrm{U(f,P)} \mid \mathrm{P\ is\ a\ partition\ of\ [a,b]}\},$$
and the lower integral of f, denoted $\underline{\int_a^b} f(x)dx$, is defined by
$$\underline{\int_a^b} f(x)dx = \mathrm{lub}\{\mathrm{L(f,P)} \mid \mathrm{P\ is\ a\ partition\ of\ [a,b]}\}.$$</td></tr>
<tr><td>Lemma 5.4.9</td><td>Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is bounded. Then the upper integral and lower integral of f always exist, and
$$\underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx.$$</td></tr>
<tr><td>Proper Integral (Theorem 5.4.10)</td><td>Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is bounded. Then f is integrable if and only if
$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx,$$
and if this equality holds then
$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$</td></tr>
<tr><td>Continuous → Integrable (Theorem 5.4.11)</td><td>Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. If f is continuous, then f is integrable.</td></tr>
</table>

Ch. 5.5 Further Properties of the Reimann Integral

Axiom / Theorem / Lemma / Definition	Description
g ∘ f is Integrable (Theorem 5.5.1)	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let D ⊆ ℝ be a set and let f: [a,b] → ℝ and g: D → ℝ be functions. Suppose that f is integrable, and that f([a,b]) ⊆ D. 1. If g is uniformly continuous and bounded, then g ∘ f is integrable. 2. If D is a non-degenerate closed bounded interval and g is continuous, then g ∘ f is integrable.
Example 5.5.2	Let f,g: [0,1] → ℝ be defined by f(x) = 1 for all x ∈ [0,1], and g(x) = (1, if x = 0 x, if x ∈ (0,1]. Then (f/g)(x) = (1, if x = 0, 1/x, if x ∈ (0,1]. We know by Example 5.2.6 (1) that f is integrable. The function g is also integrable, as can be seen by combining Exercise 5.2.6 and Exercise 5.3.3 (3). However, even though g(x) ≠ 0 for all x ∈ [0,1], the function f g is not integrable, because integrable functions are bounded by Theorem 5.3.3, and yet f g is not bounded, a fact that is evident by looking at the graph of f g, and is proved in Example 3.2.6.
Definition 5.5.3	Let A ⊆ ℝ be a set, and let f: A → ℝ be a function. The function f is **bounded away from zero** if there is some P > 0 such that |f(x)| ≥ P for all x ∈ A.
What is Integrable (Theorem 5.5.4)	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f,g: [a,b] → ℝ be functions. Suppose that f and g are integrable. 1. f^n is integrable for all n ∈ ℕ. 2. fg is integrable. 3. If g is bounded away from zero, then f/g is integrable.
Absolute Value of Integral (Theorem 5.5.5)	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. If f is integrable, then |f| is integrable and $$\left\|\int_a^b f(x)\,dx\right\| \leq \int_a^b \|f(x)\|\,dx$$
Theorem 5.5.6	Let D ⊆ C ⊆ ℝ be non-degenerate closed bounded intervals, and let f: C → ℝ be a function. If f is integrable, then $f\|_D$ is integrable.
Intermediate Bound (Theorem 5.5.7)	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let c ∈ (a,b) and let f: [a,b] → ℝ be a function. 1. f is integrable if and only if $f\|_{[a,c]}$ and $f\|_{[c,b]}$ are integrable. 2. If f is integrable, then $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Swap Bounds / Same Bounds (Definition 5.5.8)	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is integrable. Let $\int_a^b f(x)\,dx$ be defined by $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx,$$ and let $\int_a^a f(x)\,dx$ be defined by $$\int_a^a f(x)\,dx = 0$$
Split Bounds of Integration (Corollary 5.5.9)	Let C ⊆ ℝ be a closed bounded interval, and let f: C → ℝ be a function. Let a,b, c ∈ C. If f is integrable, then $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Ch. 5.6 Fundamental Theorem of Calculus

Axiom / Theorem / Lemma / Definition	Description
Example 5.6.1	(1) Let f: [0,2] → ℝ be defined by f(x) = x for all x ∈ [0,2]. Let F: [0,2] → ℝ be defined by $$F(x) = \int_1^x f(t)\,dt$$
	(2) Let h: [0,2] → ℝ be defined by h(x) = (1, if x ∈ [0,1] 2, if x ∈ (1,2].
Fundamental Theorem of Calculus Version I (Theorem 5.6.2)	Let I ⊆ ℝ be a non-degenerate interval, let a ∈ I and let f: I → ℝ be a function. Suppose that $f\|_C$ is integrable for every non-degenerate closed bounded interval C ⊆ I. Let F: I → ℝ be defined by $$F(x) = \int_a^x f(t)\,dt$$ for all x ∈ I. Let c ∈ I. If f is continuous at c, then F is differentiable at c and F'(c) = f(c). If f is continuous, then F is differentiable and F' = f.
Continuous → Antiderivative (Corollary 5.6.3)	Let I ⊆ ℝ be a non-degenerate interval, and let f: I → ℝ be a function. If f is continuous, then f has an antiderivative.
Fundamental Theorem of Calculus Version II (Theorem 5.6.4)	Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is integrable and f has an antiderivative. If F: [a,b] → ℝ is an antiderivative of f, then $$\int_a^b f(x)\,dx = F(b) - F(a)$$
Example 5.6.5	(1) Let f: ℝ → ℝ be defined by f(x) = (x^2 sin 1/x^2, if x ≠ 0 0, if x = 0.
	(2) Let h: [0,2] → ℝ be defined by h(x) = (1, if x ∈ [0,1], 2, if x ∈ (1,2].
Example 5.6.6	(1) Let g: [0,2] → ℝ be defined by g(x) = x^2 for all x ∈ [0,2].
	(2) $\int_{-1}^{1} \frac{1}{x^2}\,dx$

Sources:

- SNHU MAT 260 - Cryptology, Invitation to Cryptology, 1st Edition, Thomas Barr, 2001.
- SNHU MAT 470 - Real Analysis, The Real Numbers and Real Analysis, Ethan D. Bloch, Springer New York, 2011.