

Harold's Partial Fraction Decomposition (Calculus)

Cheat Sheet

7 April 2024

Partial Fractions	http://en.wikipedia.org/wiki/Partial_fraction_decomposition
Condition	$f(x) = \frac{P(x)}{Q(x)} = \frac{ax^n + \dots + b}{cx^m + \dots + d}$ where $P(x)$ and $Q(x)$ are polynomials
Preparation	Case 1: $n \geq m$, Perform long division first Case 2: $n < m$, Proceed to the cases below
Case I: Simple linear (1 st degree)	$\frac{A}{(ax + b)}$ or $\frac{A}{x}$
Case II: Multiple degree linear (1 st degree)	$\frac{A}{(ax + b)} + \frac{B}{(ax + b)^2} + \frac{C}{(ax + b)^3}$
Case III: Simple quadratic (2 nd degree)	$\frac{Ax + B}{(ax^2 + bx + c)}$
Case IV: Multiple degree quadratic (2 nd degree)	$\frac{Ax + B}{(ax^2 + bx + c)} + \frac{Cx + D}{(ax^2 + bx + c)^2} + \frac{Ex + F}{(ax^2 + bx + c)^3}$
Example Expansion	$\frac{P(x)}{(ax + b)(cx + d)^2(ex^2 + fx + g)}$ $= \frac{A}{(ax + b)} + \frac{B}{(cx + d)} + \frac{C}{(cx + d)^2} + \frac{Dx + E}{(ex^2 + fx + g)}$
Typical Solution for Cases I & II	$\int \frac{a}{x + b} dx = a \ln x + b + C$
Typical Solution for Cases III & IV	$\int \frac{a}{x^2 + b^2} dx = \frac{a}{b} \tan^{-1}\left(\frac{x}{b}\right) + C$

Steps to Solve	Calculus Example
1. Write down problem	$\int \frac{5x + 1}{2x^2 - x - 1} dx$
2. Check if long division is needed	Not needed since degree of numerator (top) is less than degree of denominator (bottom)
3. Factor the denominator	$\frac{5x + 1}{(2x + 1)(x - 1)}$
4. Expand function with A, B, Cs	$\frac{5x + 1}{(2x + 1)(x - 1)} = \frac{A}{(2x + 1)} + \frac{B}{(x - 1)}$
5. Find a common denominator	$= \frac{A(x - 1)}{(2x + 1)(x - 1)} + \frac{B(2x + 1)}{(2x + 1)(x - 1)}$

6. Focus on numerator	$5x + 1 = A(x - 1) + B(2x + 1)$
7. FOIL if necessary	$(x + 1)(x - 2) = x^2 - x - 2$
8. Expand/distribute the A, B, Cs	$5x + 1 = Ax - A + 2Bx + B$
9. Regroup by powers of x. (x^2, x, c)	$5x + 1 = Ax + 2Bx - A + B$
10. Factor by powers of x. $()x^2 + ()x + (c)$	$(5)x + (1) = (A + 2B)x + (-A + B)$
11. Introduce ghost factors if needed $(0, 1)$	$5x + 1 = (0)x^2 + (5)x + (1)$
12. Match left and right coefficients for a system of equations	$A + 2B = 5$ $-A + B = 1$
13. Solve system of equations	Pick simplest method below
a. Substitution method	$B = A + 1$ $A + 2(A + 1) = 5$ $A + 2A + 2 = 5$ $3A = 3$ $A = 1$ $B = A + 1 = 1 + 1 = 2$ $A = 1, B = 2$
b. Row elimination method	$A + 2B = 5$ $+ [-A + B = 1]$ ----- $3B = 6$ $B = 2$ $A + 2B = 5$ $-2 [-A + B = 1]$ ----- $3A = 3$ $A = 1$
c. Augmented matrix method	$\begin{bmatrix} A & B & & k \\ A & B & & k \end{bmatrix} = \begin{bmatrix} 1 & 2 & & 5 \\ -1 & 1 & & 1 \end{bmatrix}$ Use TI-84 <i>rref()</i> function $= \begin{bmatrix} 1 & 0 & & 1 \\ 0 & 1 & & 2 \end{bmatrix}$ $A = 1, B = 2$
14. Reassemble newly expanded function	$\frac{5x + 1}{(2x + 1)(x - 1)} = \frac{1}{2x + 1} + \frac{2}{x - 1}$
15. Verify function for accuracy	Verify the two equations are the same by plugging in any value for x and see if f(x) is the same for both.
16. Restate problem with expanded function	$\int \frac{1}{2x + 1} dx + \int \frac{2}{x - 1} dx$
17. Integrate restated problem	$\int \frac{1}{2x + 1} dx$ $u = 2x + 1$ $du = 2 dx$ $\frac{1}{2} \int \frac{1}{u} 2 dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln u = \frac{1}{2} \ln 2x + 1 $ $= \frac{1}{2} \ln 2x + 1 + 2 \ln x - 1 + C$
18. Simplify	$= \ln \sqrt{ 2x + 1 } + \ln(x - 1)^2 + C$

19. DONE

$$= \ln \left[\sqrt{|2x + 1|} (x - 1)^2 \right] + C$$