Harold's Logic (Philosophy)
Cheat Sheet
1 October 2025

The 7 Basic Logical Symbols

Operator	Symbol	Example	English
1) Intersection (AND)	∩, •	A • B	• Conjunction • A and B • A, but B • despite the fact that A, B • even though A, B • although A, B • overlap
2) Union (OR)	∪, ∨	A ∨ B	• Disjunction • A or B • inclusive or • both combined
3) Negation (NOT)	~, $\bar{A}$	~A	• not A
4) Conditional	→, ⊃	A ⊃ B	• if A then q • if A, B • B if A • A implies B • A only if B • B in case that A • A is sufficient for B • B is necessary for A
5) Biconditional	↔, ⟷, ⬄, ⇔, ⟺	p ⟷ q	• A iff B • A if and only if B • A is necessary and sufficient for B • if A then B, and conversely • if not A then not B, and conversely
6) Universal Quantifier	(x), $\forall x$	$(x)\ p(x)$	• for all • for any • for each
7) Existential Quantifier	$(\exists x)$	$(\exists x)\ p(x)$	• there exists • there is at least one
Equivalence (See Biconditional)	≡	expression$_1$ ≡ expression$_2$	• is identical to • is equivalent to • is defined as • the two expressions always have the same truth value

"… the structure of all mathematical statements can be understood using these symbols, and all mathematical reasoning can be analyzed in terms of the proper use of these symbols."
Source: "How to Prove It: A Structured Approach", 3rd Edition, p. 75.

Logical Truth Tables

A	B	AND •	NOT AND ~•	OR ∨	NOT OR ~∨	XOR ⊻, ⊕	NOT XOR ⊙	NOT ~ (~A)	If … Then ⊃	Iff ≡	Taut-ology (True) T	Contra-diction (False) F
F	**F**	F	T	F	T	F	T	T	T	T	T	F
F	**T**	F	T	T	F	T	F	T	T	F	T	F
T	**F**	F	T	T	F	T	F	F	F	F	T	F
T	**T**	T	F	T	F	F	T	f	T	T	T	F

Blank Truth Tables

Inputs				Output		
A	**B**	**C**	**D**	**X**	**Y**	**Z**
F	**F**	**F**	**F**			
F	**F**	**F**	**T**			
F	**F**	**T**	**F**			
F	**F**	**T**	**T**			
F	**T**	**F**	**F**			
F	**T**	**F**	**T**			
F	**T**	**T**	**F**			
F	**T**	**T**	**T**			
T	**F**	**F**	**F**			
T	**F**	**F**	**T**			
T	**F**	**T**	**F**			
T	**F**	**T**	**T**			
T	**T**	**F**	**F**			
T	**T**	**F**	**T**			
T	**T**	**T**	**F**			
T	**T**	**T**	**T**			

Inputs			Output	
A	**B**	**C**	**X**	**Y**
F	**F**	**F**		
F	**F**	**T**		
F	**T**	**F**		
F	**T**	**T**		
T	**F**	**F**		
T	**F**	**T**		
T	**T**	**F**		
T	**T**	**T**		

Inputs		Output
A	**B**	**X**
F	**F**	
F	**T**	
T	**F**	
T	**T**	

Precedence Rules (PEMDAS for Logic)

#	Operator	Symbol	Precedence
1	**Parenthesis**	( )	Highest precedence
2	**NOT**	~	
3	**Quantifiers**	(x), (∃x)	
4	**AND**	•	Applied Left to Right
5	**OR**	∨	
6	**Conditional**	⊃	
7	**Biconditional**	≡	Lowest precedence

Logical Conditional Connective Laws

Law or Statement	Logical Expression	Is Equivalent To (≡)	Description
Conditional Laws	$p \supset q$	$\sim p \vee q$ $\sim(p \bullet \sim q)$ Logical Equivalences: $p \vee q \equiv \sim p \supset q$ $p \bullet q \equiv \sim(p \supset \sim q)$ $\sim(p \supset q) \equiv p \bullet \sim q$ $(p \supset q) \bullet (p \supset r) \equiv p \supset (q \bullet r)$ $(p \supset q) \vee (p \supset r) \equiv p \supset (q \vee r)$ $(p \supset r) \bullet (q \supset r) \equiv (p \bullet q) \supset r$ $(p \supset r) \vee (q \supset r) \equiv (p \vee q) \supset r$	Conditional, If ... Then, Implication
Biconditional Laws (Equivalence)	$p \equiv q$ $p \leftrightarrow q$	$(p \supset q) \bullet (q \supset p)$ $(p \supset q) \bullet (\sim p \supset \sim q)$ $(p \bullet q) \vee (\sim p \bullet \sim q)$ $\sim p \leftrightarrow \sim q$ Logical Equivalences: $\sim (p \leftrightarrow q) \equiv p \leftrightarrow \sim q$	Bi-conditional, If and only If, iff, XNOR Is equivalent to
Converse*	$p \supset q$	$\not\equiv q \supset p$	False
Inverse*	$p \supset q$	$\not\equiv \sim p \supset \sim q$	False

Rules of Implication
(Inference with Propositions)

Rule Name	Rule Logic	Example
Hypothesis	Givens. First lines of a proof.	It is raining today. You live in McKinney, Texas.
Therefore	$\therefore$	Therefore. In conclusion.
1) Modus Ponens (MP)	p $\underline{p \supset q}$ $\therefore q$	It is raining today. If it is raining today, I will not ride my bike to school. Therefore, I will not ride my bike to school.
2) Modus Tollens (MT)	$\sim q$ $\underline{p \supset q}$ $\therefore \sim p$	If Sam studied for his test, then Sam passed his test. Sam did not pass his test. Therefore, Sam did not study for his test.
3) Hypothetical Syllogism (HS) (Transitivity)	$p \supset q$ $\underline{q \supset r}$ $\therefore p \supset r$	If you are mad, then you will yell. If you yell, then you will wake the baby. Therefore, if you are mad, then you will wake the baby.
4) Disjunctive Syllogism (DS) (Elimination)	$p \vee q$ $\underline{\sim p}$ $\therefore q$	Sam studied for his test or Sam took a nap. Sam did not study for his test. Therefore, Sam took a nap.
5) Constructive Dilemma (CD)	$p \vee q$ $\underline{(p \supset r) \bullet (q \supset s)}$ $\therefore r \vee s$	Oscar is either a dog or a cat. If Oscar is a dog, then you'll have fleas, and if Oscar is a cat, then you'll have fur balls. Therefore, you'll have either fleas or fur balls.
6) Simplification (Simp) (Specialization)	$\underline{p \bullet q}$ $\therefore p$	It is rainy today and it is windy today. Therefore, it is rainy today.
7) Conjunction (Conj)	p $\underline{q}$ $\therefore p \bullet q$	Sam studied for his test. Sam passed his test. Sam studied for his test and passed his test.
8) Addition (Add) (Generalization)	$\underline{p}$ $\therefore p \vee q$	It is raining today. Therefore, it is either raining today or snowing today or both.
Resolution	$p \vee q$ $\underline{\sim p \vee q}$ $\therefore q \vee r$	Your shirt is red or your pants are blue. Your shirt is not red or your pants are blue. Therefore, your pants are blue or your shoes are white.

Proof by Division into Cases	$p \lor q$ $p \supset r$ $\underline{q \supset r}$ $\therefore r$	It is raining or it is Monday. It is raining, so it is wet. It is Monday, so it is wet. It is wet.
Contradiction Rule	$\underline{\sim p \supset F}$ $\therefore p$	If it is not raining is a false statement; then it is raining.

Rules of Replacement
(Logical Connective Laws / Equivalences)

Law	Or Example	And Example
9. De Morgan's rule (DM) (Propositional Logic)	p ∨ q ≡ ~(~p • ~q) ~(p ∨ q) ≡ ~p • ~q (p ∨ ~q) ⊃ r ≡ ~r ⊃ (~p • q)	p • q ≡ ~ (~p ∨ ~q) ~(p • q) ≡ ~p ∨ ~q (p • ~q) ⊃ r ≡ ~r ⊃ (~p ∨ q)
10. Commutative (Com)	p ∨ q ≡ q ∨ p	p • q ≡ q • p
11. Associative (Assoc)	(p ∨ q) ∨ r ≡ p ∨ (q ∨ r)	(p • q) • r ≡ p • (q • r)
12. Distributive (Dist)	p • (q ∨ r) ≡ (p • q) ∨ (p • r)	p ∨ (q • r) ≡ (p ∨ q) • (p ∨ r)
13. Double Negations (DN) (Involution Law)	~ ~p ≡ p	
14. Transposition (Trans) (Contrapositive)	(p ⊃ q) ≡ (~q ⊃ ~p)	
15. Material Implication (Impl)	(p ⊃ q) ≡ (~p ∨ q)	
16. Material Equivalence (Equiv)	(p ≡ q) ≡ [(p • q) ∨ (~p • ∼q)]	(p ≡ q) ≡ [(p ⊃ q) • (q ⊃ p)]
17. Exportation (Exp)	[(p ∨ q) ⊃ r] ≡ [(p ⊃ r) ∨ (q ⊃ r)]	[(p • q) ⊃ r] ≡ [p ⊃ (q ⊃ r)]
18. Tautology (Taut) (Idempotent)	p ≡ (p ∨ p)	p ≡ (p • p)
Contradiction (Identity)	p ∨ F ≡ p	p • T ≡ p
Domination, Null (Universal Bound Laws)	p ∨ T ≡ T	p • F ≡ F
Negation, Complement (Complementary Laws)	p ∨ ~p ≡ T ~F ≡ T	p • ~p ≡ F ~T ≡ F
Uniting	(p • q) ∨ (p • ~q) ≡ p	(p ∨ q) • (p ∨ ~q) ≡ p
Absorption	p ∨ (p • q) ≡ p	p • (p ∨ q) ≡ p
Multiplying and Factoring Laws	(p ∨ q) • (~p ∨ r) ≡ (p • r) ∨ (~p • q)	(p • q) ∨ (~p • r) ≡ (p ∨ r) • (~p ∨ q)
Consensus	(p • q) ∨ (q • r) ∨ (~p • r) ≡ (p • q) ∨ (~p • r)	(p ∨ q) • (q ∨ r) • (~p ∨ r) ≡ (p ∨ q) • (~p ∨ r)
Exclusive Or (⊕)	p ⊕ q ≡ (p ∨ q) ∨ ~(p • q)	p ⊕ q ≡ (~p • q) ∨ (p ∨ ~q)

Proof Methods

<table>
<tr><th>Method</th><th>Definition</th></tr>
<tr><td>Direct</td><td>The conclusion is established by logically combining the axioms, definitions, and earlier theorems.
When given $P \supset Q$, assume P is true, then prove Q.</td></tr>
<tr><td>Indirect
(Contradiction)</td><td>If some statement is assumed true, and a logical contradiction occurs, then the statement must be false.
Or assume that the theorem is false and then show that some logical inconsistency arises as a result of the assumption, such as $r \bullet \sim r$.
Indirect proof.
Can also be a proof by counterexample. E.g., Assume $\sim(p \supset q)$, which is equivalent to $p \bullet \sim q$.</td></tr>
<tr><td>Conditional</td><td>A conditional proof is a structured argument that assumes the antecedent (p) of a conditional statement and then shows that this assumption logically leads to the consequent (q).
The goal is not to prove p is true in reality, but to prove that if p were true, then q would necessarily follow.</td></tr>
<tr><td>Contrapositive</td><td>Infers the statement $p \supset q$ by establishing the logically equivalent contrapositive statement: $\neg q \supset \sim p$.
When given $p \supset q$, assume $\sim q$ is true, then prove $\sim p$.
We prove that if the negation of the original conclusion is false, then the negation of the initial theorem is false.
Relies on De Morgen's Law.
Modus tollens.
<table>
<tr><th>p</th><th>q</th><th>If $\supset$ Then</th><th>Technique</th></tr>
<tr><td>F</td><td>F</td><td>T</td><td>Modus Tollens</td></tr>
<tr><td>F</td><td>T</td><td>T</td><td></td></tr>
<tr><td>T</td><td>F</td><td>F</td><td></td></tr>
<tr><td>T</td><td>T</td><td>T</td><td>Modus Ponens</td></tr>
</table>
A proof by contrapositive is a special case of a proof by contradiction (indirect).</td></tr>
<tr><td>Construction</td><td>The construction of a concrete example with a property to show that something having that property exists.
AKA proof by example.</td></tr>
<tr><td>Exhaustion /
By Cases</td><td>The conclusion is established by dividing it into a finite number of cases and proving each one separately.</td></tr>
<tr><td>Induction</td><td>A single "base case" is proved, and an "induction rule" is proved that establishes that any arbitrary case implies the next case.</td></tr>
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Logical Quantifiers

Definition	Logical Expression	Is Equivalent To (≡)	Plain English
Universal Quantifier (x)	$(x)\ P(x)$ $(x) \in P(x)$ $(x) \in \mathbb{D},\ P(x)$ *(x), if x is in* $\mathbb{D}$ *then P(x)*	"For all *x* in the domain, *P(x)* is true" $(x) \in A\ P(x) \equiv (x)\ (x \in A \supset P(x))$ For the finite set domain of discourse $\{a_1, a_2, \ldots, a_k\}$, $(x)\ P(x) \equiv P(a_1) \bullet P(a_2) \bullet \ldots \bullet P(a_k)$	• for all • all elements • for each member • any • every • everyone • everybody • everything • x could be anything at all
Existential Quantifier (∃x)	$(\exists x)\ P(x)$ $(\exists x) \in P(x)$ $(\exists x) \in \mathbb{D},\ P(x)$	"There exists x in the domain, such that *P(x)* is true" For the finite set domain of discourse $\{a_1, a_2, \ldots, a_k\}$, $(\exists x)\ P(x) \equiv P(a_1) \vee P(a_2) \vee \ldots \vee P(a_k)$ $P(x) \neq \emptyset$	• there exists an x • there is • some • someone • somebody • at least one value of x • there is at least one x • it is the case that • the truth set is not equal to ∅
Uniqueness Quantifier (∃!)	$\exists!x\ P(x)$	there is a unique x in *P(x)* such that … $(\exists x)\ (P(x) \bullet \sim(y)\ (P(y) \bullet y \neq x))$ $(\exists x)\ (P(x) \bullet (y)\ (P(y) \supset y = x))$ $(\exists x)\ (y)\ (P(y) \equiv y = x)$ $(\exists x)\ P(x) \bullet (y)\ (z)\ ((P(y) \bullet P(z)) \supset y = z)$	• unique • there is a unique x • there exists exactly one • there is exactly one x such that *P(x)*
Negated Existential Quantifier	$\sim [(\exists x)\ P(x)]$	$(x) \sim P(x)$	• nobody • no one • not one • there does not exist
	$\sim [(x)\ P(x)]$	$(\exists x) \sim P(x)$	

Rules of Inference with Quantifiers

Rule Name	Rule Logic	Example
Variables	**x** : Quantified variable	The domain is the set of all integers.
Elements	**c, d** : Elements of the domain, arbitrary or particular	c is a particular integer. Element definition.
Universal Instantiation	c is an element (arbitrary or particular) (x) P(x) ∴ P(c)	Sam is a student in the class. Every student in the class completed the assignment. Therefore, Sam completed his assignment.
Universal Generalization	c is an arbitrary element P(c) ∴(x) P(x)	Let c be an arbitrary integer. $c \leq c^2$ Therefore, every integer is less than or equal to its square.
Existential Instantiation*	(∃x) P(x) ∴ (c is a particular element) • P(c)	There is an integer that is equal to its square. Therefore, $c^2 = c$, for some integer c. i.e., If an object is known to exist, then that object can be given a name.
Existential Generalization	c is an element (arbitrary or particular) P(c) ∴(∃x) P(x)	Sam is a particular student in the class. Sam completed the assignment. Therefore, there is a student in the class who completed the assignment.

Quantifier Laws

Definition	Logical Expression	Is Equivalent To (≡)	Plain English
Abbreviation	$(\exists x)\ (x \in A \bullet {\sim}P(x))$	$(\exists x) \in A\ {\sim}P(x)$	Simplification
Expanding Abbreviation	$(x) \in A\ P(x)$	$(x)\ (x \in A \supset P(x))$	Complication
Quantifier Negation Laws	$(x)\ {\sim}P(x)$	${\sim}(\exists x)\ P(x)$	• nobody's perfect
	${\sim}(x)\ P(x)$	$(\exists x)\ {\sim}P(x)$	• not everyone is perfect • someone is imperfect
Conditional Law	$x \in A \supset P(x)$	$x \notin A \lor P(x)$	$p \supset q \equiv {\sim}p \lor q$
Subset Negation Law	$x \in A$	${\sim}(x \notin A)$	Swap ∈ with ∉, or vice versa
De Morgan's Law (Quantifier Negation)	${\sim}(x)\ P(x) \equiv (\exists x)\ {\sim}P(x)$ ${\sim}(\exists x)\ P(x) \equiv (x)\ {\sim}P(x)$ ${\sim}(x)\ (y)\ P(x, y) \equiv (\exists x)\ (\exists y)\ {\sim}P(x, y)$ ${\sim}(x)\ (\exists x)\ P(x, y) \equiv (\exists x)\ (y)\ {\sim}P(x, y)$ ${\sim}(\exists x)\ (y)\ P(x, y) \equiv (x)\ (\exists y)\ {\sim}P(x, y)$ ${\sim}(\exists x)\ (\exists y)\ P(x, y) \equiv (x)\ (y)\ {\sim}P(x, y)$		De Morgan's Law for single and nested quantifiers
Nested / Multiple-Quantified Statements	$(x)\ (y)$	$(y)\ (x)$	• for all objects x and y, …
	$(\exists x)\ (\exists y)$	$(\exists y)\ (\exists x)$	• there are objects x and y such that …
	$(x)\ (\exists y)\ P(x, y) \not\equiv (\exists x)\ (y)\ P(x, y)$		False Counterexample for $x, y \in \mathbb{Z}$: $(x)\ (\exists y)\ (x + y = 0) \equiv$ True $(\exists x)\ (y)\ (x + y = 0) \equiv$ False
	${\sim}((x)\ (\exists y)\ P(x, y))$	$(\exists x)\ (y)\ {\sim}P(x, y)$	Negation of multiply-quantified statements
	${\sim}((\exists x)\ (y)\ P(x, y))$	$(x)\ (\exists y)\ {\sim}P(x, y)$	
Moving Quantifiers	$(x)\ (P(x) \supset (\exists y)\ Q(x, y)) \equiv$ $(x)\ (\exists y)\ (P(x) \supset Q(x, y))$		You can move a quantifier left if the variable is not used yet

Quantifier Logic Examples

Action	Logical Statement	Plain English
Everyone	$(x)\ (y)\ P(x, y)$ NOTE: includes (x = y)	• everyone <did something> to everyone
Everyone Else	$(x)\ (y)\ (x \neq y) \supset P(x, y)$ NOTE: excludes (x = y)	• everyone <did something> to everyone else
Someone Else	$(x)\ (\exists y)\ ((x \neq y) \bullet P(x, y))$ NOTE: excludes (x = y)	• everyone <did something> to someone else
Exactly One	$(\exists x)\ (P(x) \bullet (y)\ ((x \neq y) \supset {\sim}P(y))) \equiv$ $\exists!x\ P(x)$	• exactly one person <did something>
No One	${\sim}(\exists x)\ P(x)$	• no one <did something>

Valid Quantifier Formulas

A		B
$(x)\ (P(x) \bullet Q(x))$	≡	$((x)\ P(x) \bullet (x)\ Q(x))$
$(\exists x)\ (P(x) \bullet Q(x))$	→	$((\exists x)\ P(x) \bullet (\exists x)\ Q(x))$
$(x)\ (P(x) \vee Q(x))$	←	$((x)\ P(x) \vee (x)\ Q(x))$
$(\exists x)\ (P(x) \vee Q(x))$	≡	$((\exists x)\ P(x) \vee (\exists x)\ Q(x))$
$(x)\ (P(x) \supset Q(x))$	←	$((\exists x)\ P(x) \supset (x)\ Q(x))$
$(\exists x)\ (P(x) \supset Q(x))$	≡	$((x)\ P(x) \supset (\exists x)\ Q(x))$
$(x)\ {\sim}P(x)$	≡	${\sim}(\exists x)\ P(x)$
$(\exists x)\ {\sim}P(x)$	≡	${\sim}(x)\ P(x)$
$(x)\ (\exists y)\ T(x, y)$	←	$(\exists y)\ (x)\ T(x, y)$
$(x)\ (y)\ T(x, y)$	≡	$(y)\ (x)\ T(x, y)$
$(\exists x)\ (\exists y)\ T(x, y)$	≡	$(\exists y)\ (\exists x)\ T(x, y)$
$(x)\ (P(x) \vee R)$	≡	$((x)\ P(x) \vee R)$
$(\exists x)\ (P(x) \bullet R)$	≡	$((\exists x)\ P(x) \bullet R)$
$(x)\ (P(x) \supset R)$	≡	$((\exists x)\ P(x) \supset R)$
$(\exists x)\ (P(x) \supset R)$	→	$((x)\ P(x) \supset R)$
$(x)\ (R \supset Q(x))$	≡	$(R \supset (x)\ Q(x))$
$(\exists x)\ (R \supset Q(x))$	→	$(R \supset (\exists x)\ Q(x))$
$(x)\ R$	←	R
$(\exists x)\ R$	→	R

Note: The above formulas are valid in classical first-order logic, assuming that *x* does not occur free in *R*.

Invalid Quantifier Formulas

A		B	Counterexample
$(\exists x)\ (P(x) \bullet Q(x))$	←	$((\exists x)\ P(x) \bullet (\exists x)\ Q(x))$	D = $\{a, b\}$, M = $\{P(a), Q(b)\}$
$(x)\ (P(x) \vee Q(x))$	→	$((x)\ P(x) \vee (x)\ Q(x))$	D = $\{a, b\}$, M = $\{P(a), Q(b)\}$
$(x)\ (P(x) \supset Q(x))$	→	$((\exists x)\ P(x) \supset (x)\ Q(x))$	D = $\{a, b\}$, M = $\{P(a), Q(a)\}$
$(x)\ (\exists y)\ T(x, y)$	→	$(\exists y)\ (x)\ T(x, y)$	D = $\{a, b\}$, M = $\{T(a, b), T(b, a)\}$
$(\exists x)\ (P(x) \supset R)$	←	$((x)\ P(x) \supset R)$	D = Ø, M = $\{R\}$
$(\exists x)\ (R \supset Q(x))$	←	$(R \supset (\exists x)\ Q(x))$	D = Ø, M = Ø
$(x)\ R$	→	R	D = Ø, M = Ø
$(\exists x)\ R$	←	R	D = Ø, M = $\{R\}$

Note: if empty domains are not allowed, then the last four implications above are in fact valid.

Sources

- Hurley, Patrick J. (2024). A Concise Introduction to Logic, 14th Edition, Cengage Learning, Inc.
- Wikipedia (2025).
  - https://en.wikipedia.org/wiki/List_of_logic_symbols
  - https://en.wikipedia.org/wiki/Truth_function#Table_of_binary_truth_functions

See Also

- Harold's Logic Cheat Sheet
- Harold's Logic (Philosophy) Cheat Sheet
- Harold's Sets Cheat Sheet
- Harold's Boolean Algebra Cheat Sheet
- Harold's Proofs Cheat Sheet