Harold's DiffEq Euler's Method Example

26 January 2022

Problem:

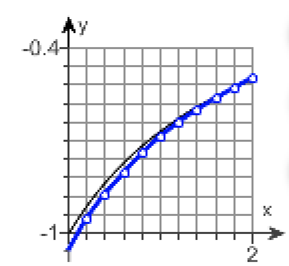
Use Euler's Method with h = 0.1 to approximate the solution to the following initial value problem on the interval 1 ≤ x ≤ 2.

Compare these approximations with the actual solution $y = -\frac{1}{x}$ by graphing the polygonal-line approximation and the actual solution on the same coordinate system.

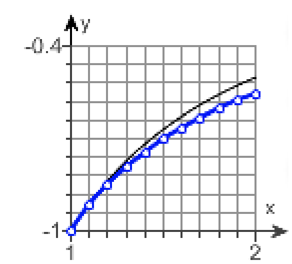
$$y' = \frac{1}{x^2} - \frac{y}{x} - y^2, \;\; y(1) = -1$$

Graph the polygonal-line approximation and the actual solution on the same coordinate system.
Choose the correct graph below.

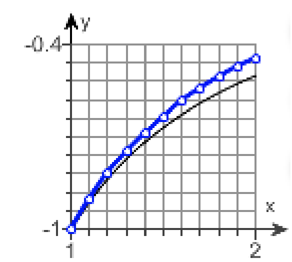
A.



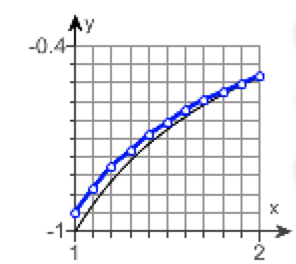
B.



C.



D.



Solution:

Euler's Method:

$\frac{dy}{dx} = f(x, y)$

$y(x_0) = y_0$

$y_{n+1} = y_n + hf(x_n, y_n)$

Notice that the 3rd equation above is simply the slope equation

$$m = \frac{\Delta y}{\Delta x} = \frac{y_{n+1} - y_n}{h} \approx f(x_n, y_n)$$

Givens:

$x_0 = 1$

$y_0 = -1$

h = 0.1

$y' = f(x, y) = \frac{1}{x^2} - \frac{y}{x} - y^2$

Step 0: (1, -1) = (x, y)

Step 1: (1.1, -0.9)

$$x_1 = x_0 + h = 1 + 0.1 = 1.1$$
$$f(x_0, y_0) = \frac{1}{(x_0)^2} - \frac{y_0}{x_0} - (y_0)^2$$
$$= f(1, -1) = \frac{1}{1^2} - \frac{-1}{1} - (-1)^2 = 1 + 1 - 1 = 1$$
$$y_1 = y_0 + hf(x_0, y_0)$$
$$y_1 = -1 + (0.1)(1) = -0.9$$

This eliminates solutions A and D.
For A, the y value is not high enough. Also, the point (1, -1) is not on the graph.
For D, x_0 should be at the bottom left corner (1,-1). It is too high.

Step 2: (1.2, -0.8240)

$$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$$
$$f(x_1, y_1) = \frac{1}{(x_1)^2} - \frac{y_1}{x_1} - (y_1)^2$$
$$= f(1.1, -0.9) = \frac{1}{(1.1)^2} - \frac{-0.9}{1.1} - (-0.9)^2 = 0.7602$$
$$y_2 = y_1 + hf(x_1, y_1)$$
$$y_2 = -0.9 + (0.1)(0.7602) = -0.8240$$

Step 3: (1.3, -0.7538)

$$x_3 = x_2 + h = 1.2 + 0.1 = 1.3$$
$$f(x_2, y_2) = \frac{1}{(x_2)^2} - \frac{y_2}{x_2} - (y_2)^2$$
$$= f(1.2, -0.8240) = \frac{1}{(1.2)^2} - \frac{-0.8240}{1.2} - (-0.8240)^2 = 0.7021$$
$$y_3 = y_2 + hf(x_2, y_2)$$
$$y_3 = -0.8240 + (0.1)(0.7021) = -0.7538$$

Step 4: (1.4, -0.6935)

$$x_4 = x_3 + h = 1.3 + 0.1 = 1.4$$
$$f(x_3, y_3) = \frac{1}{(x_3)^2} - \frac{y_3}{x_3} - (y_3)^2$$
$$= f(1.3, -0.7538) = \frac{1}{(1.3)^2} - \frac{-0.7538}{1.3} - (-0.7538)^2 = 0.6033$$
$$y_4 = y_3 + hf(x_3, y_3)$$
$$y_4 = -0.7538 + (0.1)(0.6033) = -0.6935$$

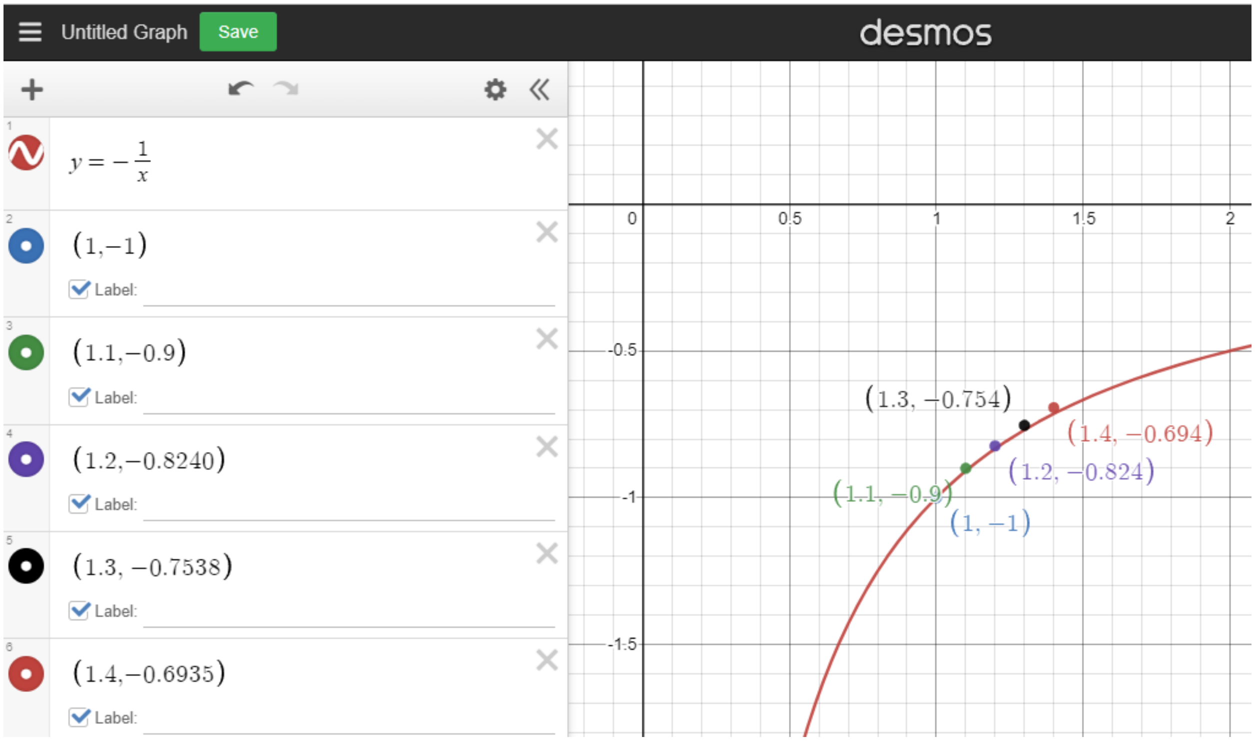
… Steps 5 – 9 …

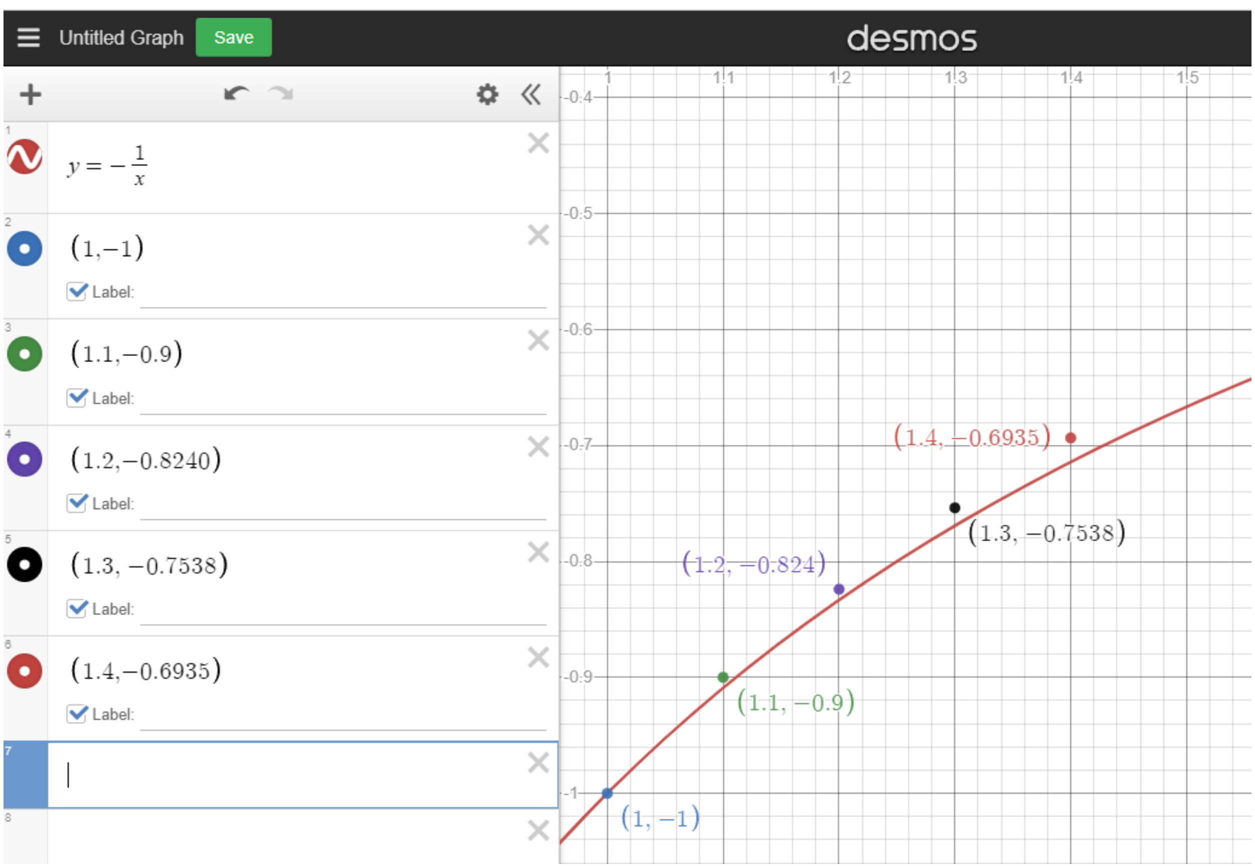
Step 9: (2.0, ?)

From the graph below, the approximation points are ABOVE the graph y = -1/x.
Since with B the approximation points are below the actual graph, the solution must be C.

Answer: C

Graph:


Untitled Graph
Save
desmos
$y = -\frac{1}{x}$
(1,−1)
Label:
(1.1,−0.9)
Label:
(1.2,−0.8240)
Label:
(1.3, −0.7538)
Label:
(1.4,−0.6935)
Label:
(1.3, −0.754)
(1.4, −0.694)
(1.2, −0.824)
(1.1, −0.9)
(1, −1)


Untitled Graph
Save
desmos
$y = -\frac{1}{x}$
(1,−1)
Label:
(1.1,−0.9)
Label:
(1.2,−0.8240)
Label:
(1.3, −0.7538)
Label:
(1.4,−0.6935)
Label:
(1.4, −0.6935)
(1.3, −0.7538)
(1.2, −0.824)
(1.1, −0.9)
(1, −1)

