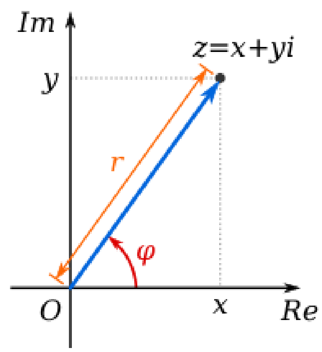
Harold's Complex Variables
Cheat Sheet
6 May 2024

Definitions

Name	Definition or Formula
Imaginary Number	$i = \sqrt{-1}$ $i^2 = -1$ $i^3 = -i$ $i^4 = 1$ i is used by mathematicians. j is used by electrical engineers.
Complex Number	 Rectangular Form (x, y): $z = x + iy$ $z = (x, y)$ where $x = \text{Re}\, z;\ y = \text{Im}\, z$ Polar Form (r, θ): $z = r(\cos\theta + i\sin\theta)$ Exponential Form (e^x): $\boldsymbol{z = re^{i\theta}}$ Parametric Form (ρ, θ): $z = z_0 + \rho e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$ Shorthand: $e^z = \exp(z) = e^x e^{iy}$
Complex Conjugate	$\bar{z} = x - iy$ $\bar{z} = r(\cos\theta - i\sin\theta)$ $\bar{z} = re^{-i\theta}$
Modulus (Magnitude/Absolute Value)	$\|z\| = \sqrt{x^2 + y^2}$ $\|z\| = r$ $\|z\|^2 = (\text{Re}\, z)^2 + (\text{Im}\, z)^2 = z \cdot \bar{z}$
Argument (Angle)	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ If $(-\pi < \theta \leq \pi)$ then principle value

Euler's Formula	$e^{i\theta} = \cos\theta + i\sin\theta$ Examples: $e^{i\frac{\pi}{2}} = i$ $e^{i\pi} = -1$ $e^{-i\frac{\pi}{2}} = -i$ $e^{i2\pi} = 1$
De Moivre's Formula	$z^n = [r\,(\cos\theta + i\sin\theta)]^n$ $\boldsymbol{z^n = r^n(\cos n\theta + i\sin n\theta)}$
Holomorphic Function (Analytic Function)	A complex variable function whose derivative exists at any point.
Meromorphic Function	A complex variable function that is holomorphic except in set points, which are poles.
Entire	A holomorphic function that is holomorphic $\forall z \in C$.
Reflection Principle	$\overline{f(z)} = f(\bar{z})$ If the lower half is the reflection of the upper half over the x-axis.

Algebraic Properties

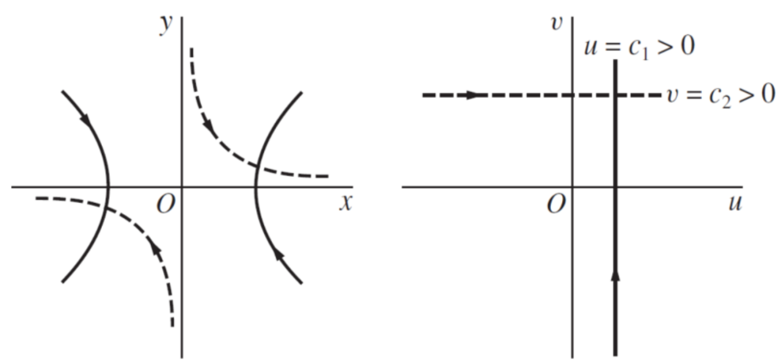
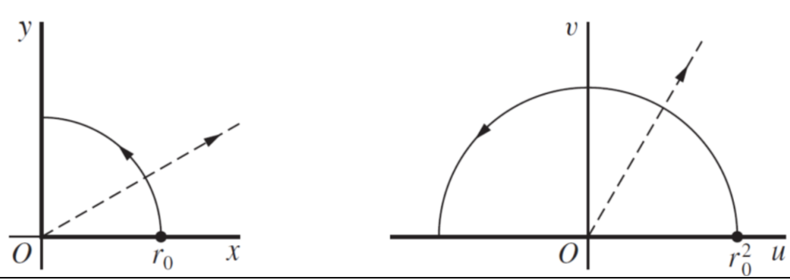
Property	Formula
Complex Numbers	$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ $z_1 - z_2 = (x_1 + x_2) - i(y_1 + y_2)$ $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$ $\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}, z_2 \neq 0$ $\lvert z_1 \cdot z_2 \cdot z_3 \cdot \ldots \cdot z_n \rvert = \lvert z_1 \rvert \cdot \lvert z_2 \rvert \cdot \lvert z_3 \rvert \cdot \ldots \cdot \lvert z_n \rvert$ $\left\lvert \frac{z_1}{z_2} \right\rvert = \frac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$
Additive Inverses	$-z = (-x, -y)$ $-z = re^{i(\theta + \pi)}$
Multiplicative Inverses	$z^{-1} = \left(\frac{x}{x^2 + y^2}, i\frac{-y}{x^2 + y^2}\right), z \neq 0$ $z^{-1} = \frac{1}{r} e^{-i\theta}$
Complex Conjugates	$\lvert \bar{z} \rvert = \lvert z \rvert$ $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$ $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$
Triangle Inequality	$\lvert z_1 \pm z_2 \rvert \leq \lvert z_1 \rvert + \lvert z_2 \rvert$ $\lvert z_1 \pm z_2 \rvert \geq \lvert z_1 \rvert - \lvert z_2 \rvert$ $\lvert z_1 + z_2 \rvert \geq \big\lvert \lvert z_1 \rvert - \lvert z_2 \rvert \big\rvert$
Exponentials	$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$ $\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$ $z^n = r^n e^{in\theta}$ $e^z = e^{z + 2\pi k i}$
Roots	$\sqrt[n]{\boldsymbol{z}} = \sqrt[n]{\boldsymbol{r}}\, \mathbf{exp}\left[\boldsymbol{i}\left(\frac{\boldsymbol{\theta}}{\boldsymbol{n}} + \frac{\mathbf{2}\boldsymbol{\pi k}}{\boldsymbol{n}}\right)\right]$
Arguments (Angles)	$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ $\arg(z_1 z_2) = (\theta_1 + \theta_2) + 2n\pi$ $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ $\arg(z_2^{-1}) = -\arg(z_2)$

Transcendental Properties

Property	Formula
Logarithms	$\log z = \ln\lvert z\rvert + i \arg z$ $\log e^z = z + 2n\pi i$ $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ $\ln\lvert z_1 z_2\rvert = \ln\lvert z_1\rvert + \ln\lvert z_2\rvert$ $\ln\lvert z_1 z_2\rvert + i\arg(z_1 z_2) = \left(\ln\lvert z_1\rvert + i\arg(z_1)\right) + \left(\ln\lvert z_2\rvert + i\arg(z_2)\right)$
Power	$z^c = e^{c \log z} = \exp\,(c \log z)$
Trigonometric	$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ $\cos z = \frac{e^{iz} + e^{-iz}}{2i}$ $\sin^2 z + \cos^2 z = 1$ $\tan^2 z + 1 = \sec^2 z$ $1 + \cot^2 z = \csc^2 z$ $\sin(2z) = 2\sin z \cos z$ $\cos(2z) = \cos^2 z - \sin^2 z$ $\cos(2z) = 2\cos^2 z - 1$ $\cos(2z) = 1 - 2\sin^2 z$
Hyperbolic	$\sinh z = \frac{e^z - e^{-z}}{2}$ $\cosh z = \frac{e^z + e^{-z}}{2}$ $\sin(ix) = i \sinh x$ $\cos(ix) = \cosh x$ $\sin z = \sin x \cosh y + i \cos x \sinh y$ $\cos z = \cos x \cosh y - i \sin x \sinh y$ $\sinh z = \sinh x \cos y + i \cosh x \sin y$ $\cosh z = \cosh x \cos y - i \sinh x \sin y$
Inverse Trigonometric	$\sin^{-1} z = -i \ln\left[iz + \sqrt{1 - z^2}\right]$ $\cos^{-1} z = -i \ln\left[z + \sqrt{z^2 - 1}\right]$ $\tan^{-1} z = \frac{i}{2} \ln\left[\frac{i + z}{i - z}\right]$ $\frac{d}{dz}\sin^{-1} z = \frac{1}{\sqrt{1 - z^2}}$ $\frac{d}{dz}\cos^{-1} z = \frac{-1}{\sqrt{1 - z^2}}$ $\frac{d}{dz}\tan^{-1} z = \frac{1}{1 + z^2}$

Inverse Hyperbolic	$\sinh^{-1} z = \ln\left[z + \sqrt{1 + z^2}\right]$ $\cosh^{-1} z = \ln\left[z + \sqrt{z^2 - 1}\right]$ $\tanh^{-1} z = \frac{1}{2} \ln\left[\frac{1+z}{1-z}\right]$

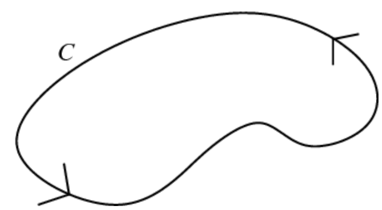
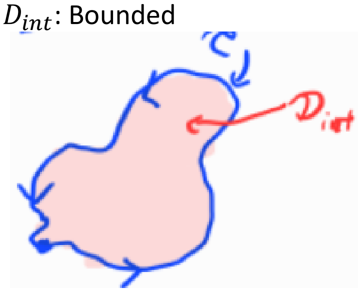
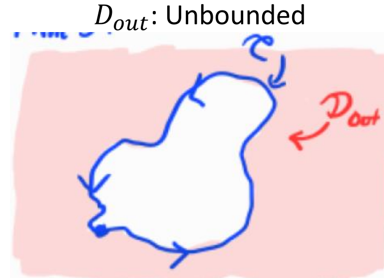
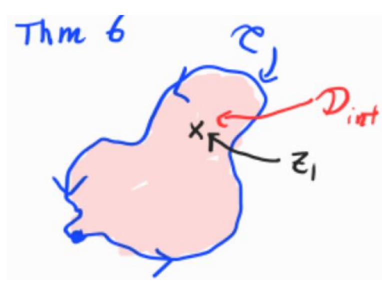
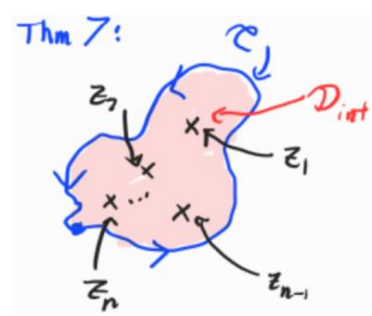
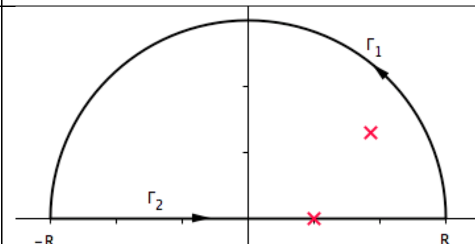
Functions

Name	Formula
Functions	$f(z) = f(x + iy) = u(x, y) + iv(x, y) = u + iv$ $f(z) = f\left(re^{i\theta}\right) = u(r, \theta) + iv(r, \theta) = u + iv$
Conic Mappings	Hyperbola (Rectangular Form): $w = z^2$ $u = x^2 + y^2 = c_1$ $v = 2xy = -2y\sqrt{y^2 + c_1}$  Circle (Polar Form): $w = z^2$ $w = r^2 e^{i2\theta}$ $\rho = r^2$ $\varphi = 2\theta$ 

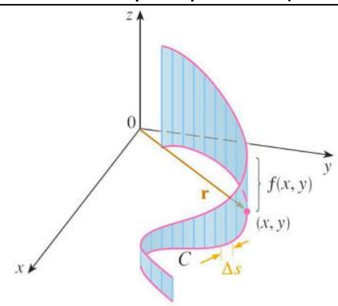
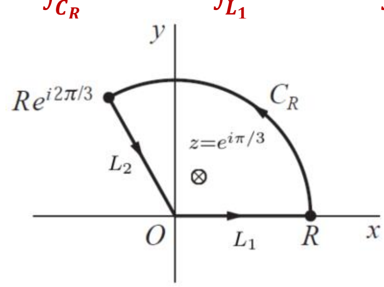
Differentiation

Name	Formula
Cauchy-Riemann Equations	Determines whether the given complex valued function $f(z) = u + iv$ is analytic and differentiable. Rectangular Form: $f(z) = u(x, y) + iv(x, y)$ and $f'(z)$ exists at point $z_0 = x_0 + iy_0$ $\boldsymbol{u_x = v_y}, \quad \boldsymbol{u_y = -v_x}$ $f'(z_0) = u_x + iv_x$ where $u_x = \frac{\partial u}{\partial x}$ Polar Form: $f(z) = u(r, \theta) + iv(r, \theta)$ $ru_r = v_\theta, \quad u_\theta = -rv_r$ $f'(z_0) = e^{-i\theta}(u_r + iv_r)$
Laplace's Equation (Harmonic)	$H_{xx}(x, y) + H_{yy}(x, y) = 0$

Contours

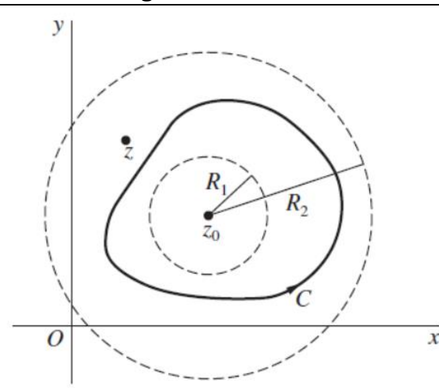
Name	Definition	
Simple Arc (C) (Jordan arc)	If arc C does not cross itself; that is, C is simple if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$. E.g., open.	
Contour (C)	A closed path in the complex plane. A piecewise smooth arc consisting of a finite number of smooth arcs joined end to end.	
Simple Curve (C)	A simple arc where $z(b) = z(a)$. E.g., closed. Defaults to the unit circle with interval $[0, 2\pi]$.	Simple closed curve 
Positively Oriented	a *simple closed curve,* or a Jordan curve is **positively** oriented when it is in the **counterclockwise** direction.	
Branch Cut	A portion of a line or curve that is introduced to define a branch F of a multiple-valued function f.	
Regions Bound by Curve C	D_{int}: Bounded 	D_{out}: Unbounded 
Closed, Simple, Counter-Clockwise Oriented Curve	One Point, Simple Pole: 	Multiple Points, Simple Poles: 
Poles	Roots in the denominator of a complex function that is holomorphic (complex differential). E.g., Singularity, asymptote.	

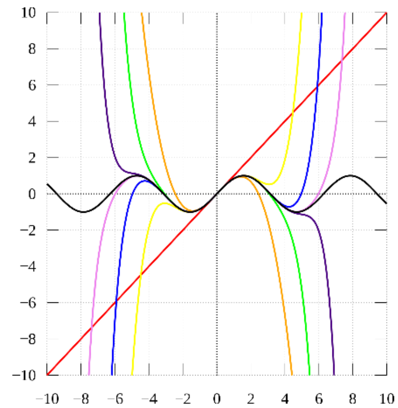
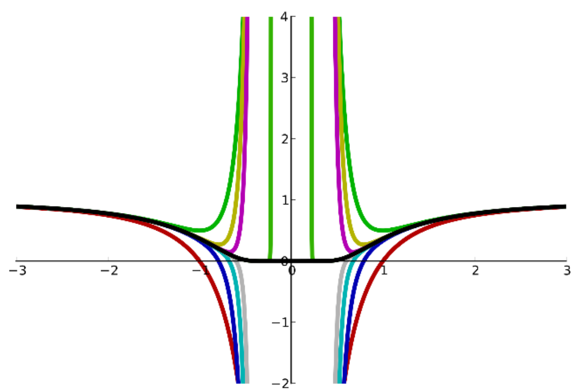
Integration

Name	Formula
Complex Integration	$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy) = \int_C udx - vdy + i\int_C vdx - udy$ *"No corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane."* (Brown & Churchill, p.125)
Contour Integral (Complex $\mathbb{C}$) **Line Integral** (Real $\mathbb{R}$)	
Contour Integral	$\boldsymbol{\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt}$
Fundamental Theorem of Calculus with Contour Integral	$\int_C f(z)\,dz = \int_a^b f\big(z(t)\big)\,z'(t)dt = F\big(z(t)\big)\Big]_a^b$ $= F(z(b)) - F(z(a)) = F(z_2) - F(z_1)$ Since $z(b) = z_2$ and $z(a) = z_1$. where z_0 is a point within the contour C.
Morera's Theorem	If $\int_C f(z)\,dz = 0$ then, f is **Holomorphic** over $\mathcal{R}$.
Cyclic Integral	$\oint_C f(z)\,dz$ Integral over a **closed** contour meaning the curve returns to its initial position ($a = b$). For circular contours, $\oint_{C_R} f(z)\,dz = \int_0^{2\pi} f(z)\,dz$ For non-circular contours, $\boldsymbol{\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz}$  Parameterize the arcs and identify the bounds of integration. $L_1: z = re^{i0}, \quad 0 \le r \le R$ $L_2: z = re^{i2\pi/3}, \quad R \le r \le 0$ $C_R: z = Re^{i\theta}, \quad 0 \le \theta \le 2\pi/3$

Cauchy-Goursat Theorem (Cauchy's Integral Theorem)	D_{int}: If C is closed, i.e., $z_0 = z_1$, then $$\oint_C f(z)\,dz = 0$$ D_{out}: Outside of closed C, at infinity (∞): If $$\lim_{z\to\infty} f(z)\,z = 0$$ Then $$\oint_C f(z)\,dz = 0$$
Cauchy Integral Formula	Simple: $$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz$$ Extension: $$\boldsymbol{f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}dz}$$ where $n = 0, 1, 2, \ldots;\ 0! = 1;\ f^{(0)}(z_0) = f(z_0)$. Turns a contour integral into a derivative.
Jordan's Lemma	Estimation Lemma: $$\left\lvert\int_C f(z)\,dz\right\rvert \le \mathrm{length}(C)\cdot\max_{z\in C}\lvert f(z)\rvert$$ Common Application: If $$\lim_{z\to\infty} f(z)\,z = 0$$ Then $$\int_{-\infty}^{\infty} f(at+b)\,a\,dt\,x = \int_L f(z)\,dz = \lim_{R\to\infty}\oint_{C_R} f(z)\,dz$$ where line $L = \{at + b\}, -\infty < t < \infty$ and $R =$ semi-circle radius along this line.

Series

Name	Formula
Liouville's Theorem	If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.
Fundamental Theorem of Algebra	$P(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0, \quad (a_n \neq 0, n \geq 1)$ $P(z) = c(z - z_n) \ldots (z - z_2)(z - z_1)$ Any polynomial of degree n has at least one zero in the complex plane. That is, there exists at least one point z_0 such that $P(z_0) = 0$.
Maximum Modulus Principle	**Theorem:** If a function f is analytic and not constant in a given domain D, then the modulus $\lvert f(z)\rvert$ has no maximum value in D. That is, there is no point z_0 in the domain such that $\lvert f(z)\rvert \leq \lvert f(z_0)\rvert$ for all points z in it.
	Corollary: Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $\lvert f(z)\rvert$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.
Complex Variable Convergence	$\lim_{n\to\infty} z_n = z$ $\text{iff } \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$ $\text{where } z_n = x_n + iy_n$
Complex Series Convergence	$\sum_{n=1}^{\infty} z_n = S$ $\text{iff } \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y$ $\text{where } S = X + iY$
Series Convergence	**Corollary 1**: If a series of complex numbers converges, the n[th] term converges to zero as n tends to infinity. **Corollary 2:** The absolute convergence of a series of complex numbers implies the convergence of that series.
Annular Domain $R_1 < \lvert z - z_0\rvert < R_2$	
Transcendental Series	See Harold's Taylor Series Cheat Sheet for a comprehensive list of the Taylor series of all transcendental functions.

Taylor Series	$f(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n$ where disk $(\lvert z - z_0 \rvert < R_0)$ $a_n = \frac{f^{(n)}(z_0)}{n!}$ where $(n = 0, 1, 2, \dots)$ Series converges to $f(z)$ when z lies in the stated open disk. If $z_0 = 0$, then **Maclaurin series**. 
Laurent Series	$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{\boldsymbol{b_n}}{\boldsymbol{(z - z_0)^n}}$ $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz$ where $(n = 0, 1, 2, \dots)$ $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{-n+1}}\,dz$ where $(n = 1, 2, 3, \dots)$ where $(R_1 < \lvert z - z_0 \rvert < R_2)$  Taylor Series Form: $f(z) = \sum_{-\infty}^{\infty} c_n(z - z_0)^n$ $c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz = \frac{f^{(n)}(z_0)}{n!}$ where $(n = 0, 1, 2, \dots)$ and $(R_1 < \lvert z - z_0 \rvert < R_2)$ If no poles, then **Taylor series**.

Power Series

Name	Formula
Power Series	$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$ where $(\|z - z_0\| < R)$
Absolute and Uniform Convergence	**Theorem 1**: If a power series converges when $z = z_1 \ (z_1 \neq z_0)$, then it is **absolutely** convergent at each point z in the open disk $\|z - z_0\| < R_1$ where $R_1 = \|z_1 - z_0\|$.
	Theorem 2: If z_1 is a point inside the circle of convergence $\|z - z_0\| = R$ of a power series then that series must be **uniformly** convergent in the closed disk $\|z - z_0\| \leq R_1$ where $R_1 = \|z_1 - z_0\|$.
Continuity of Sums	**Theorem**: A power series represents a **continuous** function $S(z)$ at each point inside its circle of convergence $\|z - z_0\| = R$.
Integration	**Theorem**: Let C denote any contour interior to the circle of convergence of the power series and let $g(z)$ be any function that is continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be **integrated** term by term over C; that is, $$\int_C g(z)\, S(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)\, (z - z_0)^n dz\,.$$
Differentiation	**Theorem**: The power series can be **differentiated** term by term. That is, at each point z interior to the circle of convergence of that series, $$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$
Leibniz's Rule for the nth Derivative	$$[f(z)\, g(z)]^{(n)} = \sum_{n=0}^{\infty} \binom{n}{k} f^{(k)}(z)\, g^{(n-k)}(z)$$ where $(n = 1, 2, \dots)$ $$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$ where $(k = 0, 1, 2, \dots, n)$
Uniqueness Representations	**Theorem 1**: If a power series converges to $f(z)$ at all points interior to some circle $\|z - z_0\| = R$, then it is the **Taylor series** expansion for f in powers of $z - z_0$.
	Theorem 2: If a series $$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$ converges to $f(z)$ at all points in some annular domain about z_0, then it is the **Laurent series** expansion for f in powers of $z - z_0$ for that domain.
Let ...	$f(z), g(z), h(z),$ and $k(z)$ all be different power series.
Multiplication	$f(z)g(z) = h(z)$
Division	$$\frac{f(z)}{g(z)} = k(z)$$

Residues and Poles

Name	Formula
Residue	$\mathbf{Res}_{z_1}(f) = \lim\limits_{z \to z_1} (z - z_1)\, f(z)$ **Observation**: Since $z_1$is a pole, then the $\mathrm{Res}_{z_1}(f)$ turns the function f into a function with a hole or hollow point. The limit makes the remaining function appear continuous.
Cauchy's Residue Theorem	One Point, Simple Pole inside Contour C: If exists $\mathrm{Res}_{z_1}(f) = \lim\limits_{z \to z_1} (z - z_1)\, f(z)$ Then $\oint_C f(z)\, dz = 2\pi i\, (\mathrm{Res}_{z_1}(f))$
	Multiple Points, Simple Poles inside Contour C: If these exist $\sum\limits_{k=1}^{n} \mathop{\mathrm{Res}}\limits_{z=z_k} f(z) =$ $Res_{z_1}(f) = \lim\limits_{z \to z_1} (z - z_1)\, f(z)$ $+ \ldots +$ $Res_{z_n}(f) = \lim\limits_{z \to z_n} (z - z_n)\, f(z)$ Then $\oint_C f(z)\, dz = 2\pi i \sum\limits_{k=1}^{n} \mathop{\mathrm{Res}}\limits_{z=z_k} f(z)$
	Special Case: if $f(z)$ is even, then $\int_{-\infty}^{\infty} f(z)\, dz = 2\pi i \sum\limits_{k=1}^{n} \mathop{\mathrm{Res}}\limits_{z=z_k} f(z)$ $\int_{0}^{\infty} f(z)\, dz = \pi i \sum\limits_{k=1}^{n} \mathop{\mathrm{Res}}\limits_{z=z_k} f(z)$
Higher-Order Poles	$R_n = \frac{1}{(n-1)!} \lim\limits_{z \to p} \left[\frac{d^{n-1}}{d_z^{n-1}} \{(z-p)^n f(z)\} \right]$ where p is a pole in the contour region.
Zeros and Poles	**Theorem 1**: Suppose that: (a) two functions p and q are analytic at a point z_0; (b) $p(z_0) \neq 0$ and q has a **zero** of order m at z_0. Then the quotient $p(z)/q(z)$ has a **pole** of order m at z_0. **Theorem 2**: Let two functions p and q be analytic at a point z_0. If $p(z_0) \neq 0, \quad q(z_0) = 0, \quad$ and $q'(z_0) \neq 0,$ then z_0 is a simple **pole** of the quotient $p(z)/q(z)$ and $\mathop{\mathrm{Res}}\limits_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$

Textbook

- NYU MATH-UY-4434: Applied Complex Variables, Complex Variables and Applications, 9th Edition, Chapters 1-7, James Ward Brown & Ruel V. Churchill, McGraw-Hill Education, 2014.

