**Harold’s Complex Variables**

**Cheat Sheet**

9 May 2024

**Definitions**

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| **Name** | **Definition or Formula** |
| **Imaginary Number** | is used by mathematicians.  is used by electrical engineers. |
| **Complex Number** | Complex number - Wikipedia  Rectangular Form :  Polar Form :  Exponential Form ():  Parametric Form :  Shorthand: |
| **Complex Conjugate** |  |
| **Modulus**  (Magnitude/Absolute Value) |  |
| **Argument**  (Angle) | If then principle value |
| **Euler’s Formula** | Examples: |
| **De Moivre’s Formula** |  |
| **Holomorphic Function**  (Analytic Function) | A complex variable function whose derivative exists at any point. |
| **Meromorphic Function** | A complex variable function that is holomorphic except in set points, which are poles. |
| **Entire** | A holomorphic function that is holomorphic . |
| **Reflection Principle** | If the lower half is the reflection of the upper half over the x-axis. |

**Algebraic Properties**

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| **Property** | **Formula** |
| **Complex Numbers** |  |
| **Additive Inverses** |  |
| **Multiplicative Inverses** |  |
| **Complex Conjugates** |  |
| **Triangle Inequality** |  |
| **Exponentials** |  |
| **Roots** |  |
| **Arguments**  (Angles) |  |

**Transcendental Properties**

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| **Property** | **Formula** |
| **Power** |  |
| **Logarithms** |  |
| **Trigonometric** |  |
| **Hyperbolic** |  |
| **Inverse Trigonometric** |  |
| **Inverse Hyperbolic** |  |

**Functions**

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| **Name** | **Formula** |
| **Functions** |  |
| **Conic Mappings** | Hyperbola (Rectangular Form):  A diagram of a graph  Description automatically generated  Circle (Polar Form):  A black and white image of a curved line  Description automatically generated |

**Differentiation**

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| **Name** | **Formula** |
| **Cauchy-Riemann Equations** | Determines whether the given complex-valued function is analytic and differentiable.  Rectangular Form:  Polar Form: |
| **Laplace’s Equation**  **(Harmonic)** |  |

**Contours**

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| **Name** | **Definition** | |
| **Simple Arc (C)**  (Jordan arc) | If arc C does not cross itself; that is, C is simple if when . E.g., open. | |
| **Contour (C)** | A closed path in the complex plane.  A piecewise smooth arc consisting of a finite number of smooth arcs joined end to end. | |
| **Simple Curve (C)** | A simple arc where . E.g., closed. | Simple closed curve  A closed curve |
| Simple closed curve C defaults to a circle, , centered at 0 with radius and interval , oriented counterclockwise. |  |
| **Positively Oriented** | a *simple closed curve,* or a Jordan curve, is **positively** oriented when it is in the **counterclockwise** direction. | |
| **Simply Connected Domain** | is a domain such that every simple closed contour within it encloses **only** points of | |
| **Branch Cut** | A portion of a line or curve that is introduced to define a  branch of a multiple-valued function . | |
| **Regions Bound by Curve C** | : Bounded | : Unbounded |
| **Closed, Simple, Counter-Clockwise Oriented Curve** | One Point, Simple Pole: | Multiple Points, Simple Poles: |

**Integration**

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| **Name** | **Formula** | |
| **Complex Integration** | “*No corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane.*” (Brown & Churchill, p.125) | |
| **Contour Integral** (Complex ) | Called a Line Integral if Real numbers.  Change of Contour direction: | Line Integral: Vector Field and Applications |
| **Fundamental Theorem of Calculus with Contour Integral** | Since and .  where is a point within the contour . | |
| **Morera’s Theorem** | If then, is **Holomorphic** over . | |
| **Cyclic Integral** | Integral over a **closed** contour meaning the curve returns to its initial position ().  For circular contours,  For non-circular contours,    Parameterize the arcs and identify the bounds of integration. | |
| **Cauchy-Goursat Theorem**  (Cauchy’s Integral Theorem) | : If C is closed, i.e., , then  : Outside of closed C, at infinity ():  If  Then | |
| **Cauchy Integral Formula** | Turns a contour integral into a derivative.  Simple:  General: | |
| **Jordan’s Lemma** | Estimation Lemma:  Common Application:  If  Then  semi-circle radius along this line. | |

**Poles and Residues**

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| **Name** | **Formula** | |
| **Poles** | Roots in the denominator of a complex function that is holomorphic (complex differential). E.g., Singularity, vertical asymptote. | complex analysis - pole on the contour using the residu theorem, what is  this formula of Plemelj? - Mathematics Stack Exchange |
| Poles are zeros in the denominator of .  Simple Pole:  High-Order Pole:  **Theorem**: If is a pole of function , then | |
| **Residue** | **Observation**: Since is a simple pole, then the turns the function into a function with a hole or hollow point. The limit makes the remaining function appear continuous. “*Remove the pole and cover the hole.*”  **Tip:** If Laurent Series centered at 0, then  the term of . | |
| **Cauchy’s Residue Theorem**  (Simple Poles) | One Point, Simple Pole inside Contour C:  If exists  Then | |
| Multiple Points, Simple Poles inside Contour C:  If these exist  + ... +  Then | |
| Special Case:  If is even, then | |
| **Residue of High-Order Poles** | General:  Works with higher-order poles.  then  Simple:  Tip: | |
| **Residue of Simple Poles**  (shortcut) | **Theorem**: Let two functions and be analytic at a point . If  then is a simple **pole** of the quotient and | |
| **Zeros and Poles** | **Theorem**: Suppose that:  (a) two functions and are analytic at a point ;  (b) and has a **zero** of order at .  Then the quotient has a **pole** of order at . | |

**Series**

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| **Name** | **Formula** |
| **Liouville’s Theorem** | If a function is entire and bounded in the complex plane, then  is constant throughout the plane. |
| **Fundamental Theorem of Algebra** | Any polynomial of degree has at least one zero in the complex plane. That is, there exists at least one point such that. |
| **Maximum Modulus Principle** | **Theorem:** If a function is analytic and not constant in a given domain , then the modulus has no maximum value in . That is, there is no point in the domain such that for all points in it. |
| **Corollary**: Suppose that a function is continuous on a closed bounded region and that it is analytic and not constant in the interior of . Then the maximum value of in , which is always reached, occurs somewhere on the boundary of and never in the interior. |
| **Complex Variable Convergence** |  |
| **Complex Series Convergence** |  |
| **Series Convergence** | **Corollary 1**: If a series of complex numbers converges, the nth term converges to zero as n tends to infinity.  **Corollary 2:** The absolute convergence of a series of complex numbers implies the convergence of that series. |
| **Annular Domain** | A diagram of a circle with a circle and lines  Description automatically generated |
| **Transcendental Series** | See [Harold’s Taylor Series Cheat Sheet](https://www.toomey.org/tutor/harolds_cheat_sheets/Harolds_Taylor_Series_Cheat_Sheet_2024.pdf) for a comprehensive list of the Maclaurin series of all transcendental functions. |
| **Taylor Series** | Series converges to when lies in the stated open disk.  If , then **Maclaurin** **series**.  undefined |
| **Laurent Series** | undefined  Taylor Series Form:  If no poles, then **Taylor** **series**. |

**Power Series**

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| **Name** | **Formula** |
| **Power Series** |  |
| **Absolute and Uniform Convergence** | **Theorem 1**: If a power series converges when  then it is **absolutely** convergent at each point in the open disk |
| **Theorem 2**: If is a point inside the circle of convergence of a power series then that series must be **uniformly** convergent in the closed disk |
| **Continuity of Sums** | **Theorem**: A power series represents a **continuous** function at each point inside its circle of convergence . |
| **Integration** | **Theorem**: Let denote any contour interior to the circle of convergence of the power series and let be any function that is continuous on . The series formed by multiplying each term of the power series by can be **integrated** term by term over ; that is, |
| **Differentiation** | **Theorem**: The power series can be **differentiated** term by term. That is, at each point interior to the circle of convergence of that series, |
| **Leibniz’s Rule for the nth Derivative** |  |
| **Uniqueness Representations** | **Theorem 1**: If a power series converges to at all points interior to some circle , then it is the **Taylor** **series** expansion for in powers of . |
| **Theorem 2**: If a series  converges to at all points in some annular domain about , then it is the **Laurent series** expansion for in powers of for that domain. |
| **Multiplication** |  |
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| **Division** |  |

**College Course**

* **Course**: [NYU MATH-UY-4434](http://bulletin.engineering.nyu.edu/preview_course_nopop.php?catoid=15&coid=37886): Applied Complex Variables.
* **Textbook**: [Complex Variables and Applications](https://www.amazon.com/Complex-Variables-Applications-Brown-Churchill/dp/0073383171/ref=sr_1_1), 9th Edition, Chapters 1-7, James Ward Brown & Ruel V. Churchill, McGraw-Hill Education, 2014.