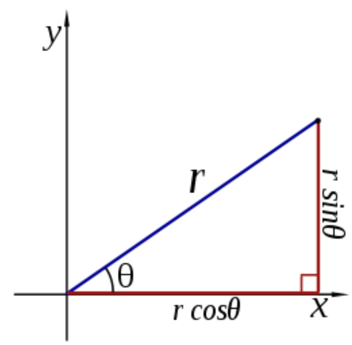
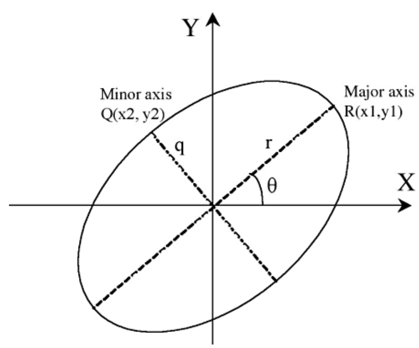
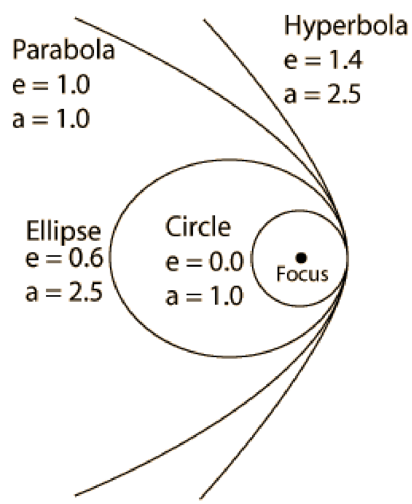
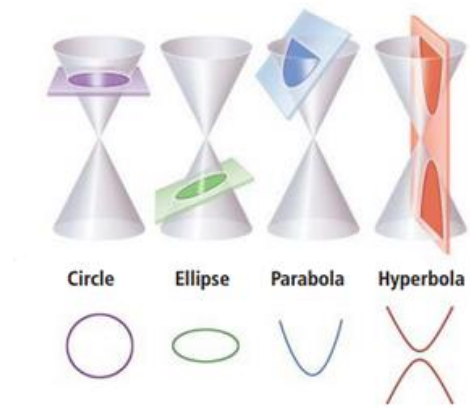
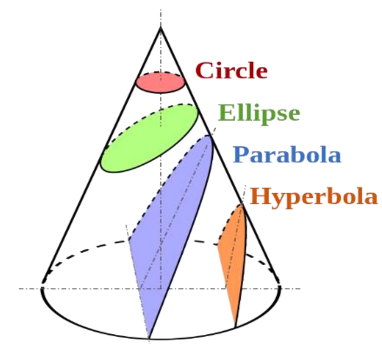
Harold's AP Calculus BC
Cheat Sheet

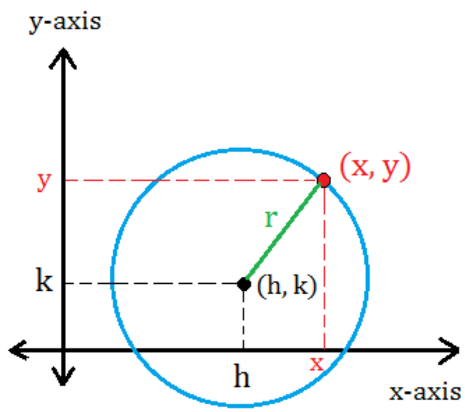
29 November 2022

	Rectangular	Polar	Parametric
Point	$f(x) = y$ (x, y) (a, b)	(r, θ) *or* $r \angle \theta$ *Polar → Rect.* $x = r\cos\theta$ $y = r\sin\theta$ $\tan\theta = \frac{y}{x}$ *Rect. → Polar* $r^2 = x^2 + y^2$ $r = \pm\sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$	*Point (a,b) in Rectangular:* $x(t) = a$ $y(t) = b$ $< a, b >$ $t = 3^{rd}$ *variable, usually time, with 1 degree of freedom (df)*
Line	*Slope-Intercept Form:* $y = mx + b$ *Point-Slope Form:* $y - y_0 = m\,(x - x_0)$ *General Form:* $Ax + By + C = 0$ *Calculus Form:* $f(x) = f'(a)\,x + f(0)$		$< x, y > = < x_0, y_0 > + t < a, b >$ $< x, y > = < x_0 + at, y_0 + bt >$ *where* $< a, b > = < x_2 - x_1,\ y_2 - y_1 >$ $x(t) = x_0 + ta$ $y(t) = y_0 + tb$ $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{b}{a}$
Plane	$n_x(x - x_0)$ $+ n_y(y - y_0)$ $+ n_z(z - z_0) = 0$	*Vector Form:* $\boldsymbol{n} \bullet (\boldsymbol{r} - \boldsymbol{r}_0) = 0$	$\boldsymbol{r} = \boldsymbol{r}_0 + s\boldsymbol{v} + t\boldsymbol{w}$ *where:* • $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *are given vectors defining the plane* • $\boldsymbol{r_0}$ *is the vector of a fixed point on the plane*

	Rectangular	Polar	Parametric
Conics	*General Equation for All Conics:* $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ *where* *Line*: $A = B = C = 0$ *Circle*: $A = C$ *and* $B = 0$ *Ellipse*: $AC > 0$ *or* $B^2 - 4AC < 0$ *Parabola*: $AC = 0$ *or* $B^2 - 4AC = 0$ *Hyperbola*: $AC < 0$ *or* $B^2 - 4AC > 0$ *Note: If* $A + C = 0$*, square hyperbola* *Rotation:* *If B ≠ 0, then rotate coordinate system:* $\cot 2\theta = \frac{A - C}{B}$ $x = x' \cos\theta - y' \sin\theta$ $y = y' \cos\theta + x' \sin\theta$ *New = (x', y'), Old = (x, y)* *rotates through angle θ from x-axis* 	*General Equation for All Conics:* $r = \frac{p}{1 - e\cos\theta}$ *where* $p = \begin{cases} a(1-e^2) \\ 2d \\ a(e^2-1) \end{cases}$ *for* $\begin{cases} 0 \le e < 1 \\ e = 1 \\ e > 1 \end{cases}$ *p = semi-latus rectum* *or the line segment running from the focus to the curve in a direction parallel to the directrix* *Eccentricity:* *Circle* $e = 0$ *Ellipse* $0 < e < 1$ *Parabola* $e = 1$ *Hyperbola* $e > 1$ 	 

<table>
<tr><th></th><th>Rectangular</th><th>Polar</th><th>Parametric</th></tr>
<tr>
<td>Circle</td>
<td>
$(x-h)^2+(y-k)^2=r^2$

Center: (h,k)
Vertices: NA
Focus: (h,k)



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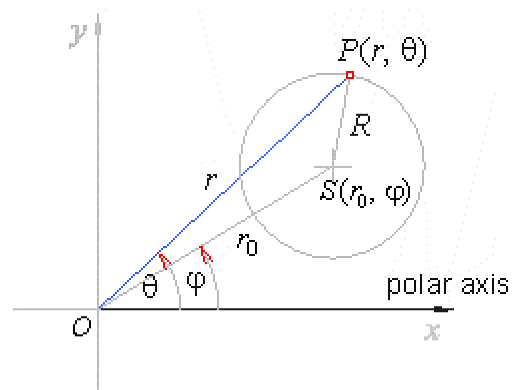
Centered at Origin:
r = a (constant)
$\theta \;=\; \theta \;\; [0,2\pi]\ or\ [0,360°]$

Centered at (r_0,ϕ):
$r^2+r_0^2-2rr_0\cos(\theta-\phi)=R^2$

Hint: Law of Cosines
or
$r=r_0\cos(\theta-\phi)$
$+\sqrt{a^2-r_0^2\ sin^2(\theta-\phi)}$



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$x(t)=r\ cos(t)+h$
$y(t)=r\ sin(t)+k$
$[t_{min},\ t_{max}]=[0,2\pi]$

Center: (h,k)
Focus: (h,k)
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<tr><th></th><th>Rectangular</th><th>Polar</th><th>Parametric</th></tr>
<tr>
<td>Ellipse</td>
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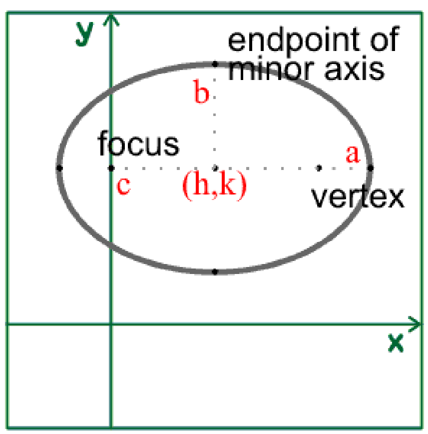
$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Center: (h, k)
Vertices: $(h \pm a, k)$ *and* $(h, k \pm b)$
Foci: $(h \pm c, k)$

Focus length, c, from center:
$c^2 = a^2 - b^2$



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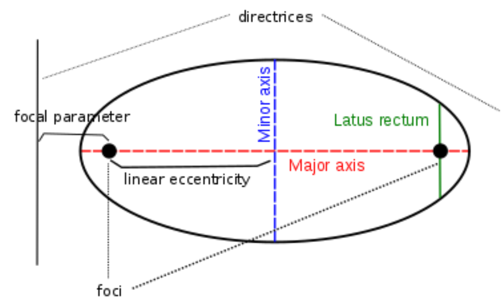
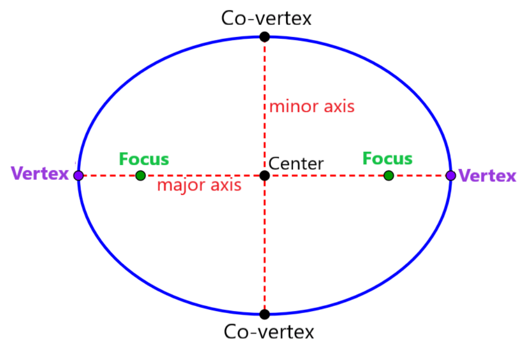
Ellipse:

$$r = \frac{a\,(1-e^2)}{1+e\cos\theta} \quad for\ 0 < e < 1$$

$$where\ e = \frac{c}{a} = \frac{\sqrt{a^2-b^2}}{a}$$

relative to center (h, k)





Interesting Note:
The <u>sum</u> of the distances from each focus to a point on the curve is constant.
$|d_1 + d_2| = k$

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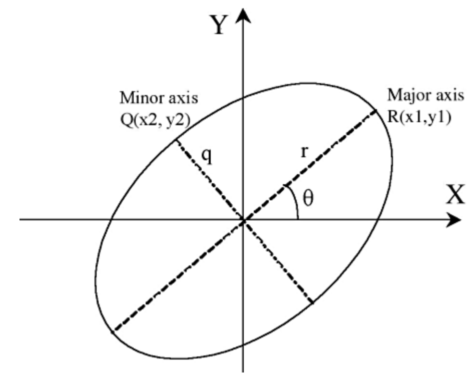
$x(t) = a\cos(t) + h$
$y(t) = b\sin(t) + k$
$[t_{min}, t_{max}] = [0, 2\pi]$

Center: (h, k)

Rotated Ellipse:
$x(t) = a\cos t\cos\theta - b\sin t\sin\theta + h$
$y(t) = a\cos t\sin\theta + b\sin t\cos\theta + k$

θ *= the angle between the x-axis and the major axis of the ellipse*



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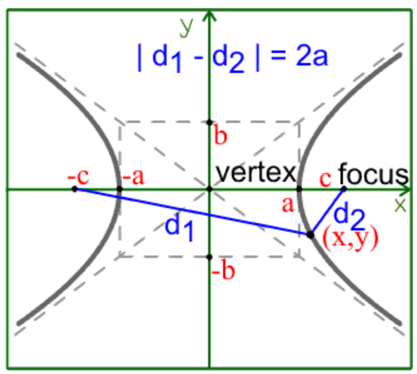
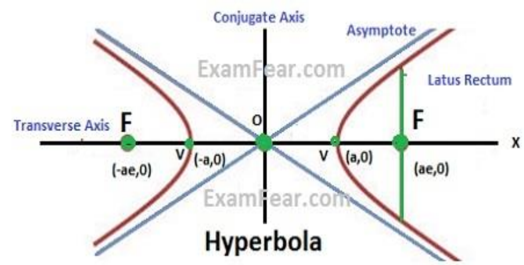
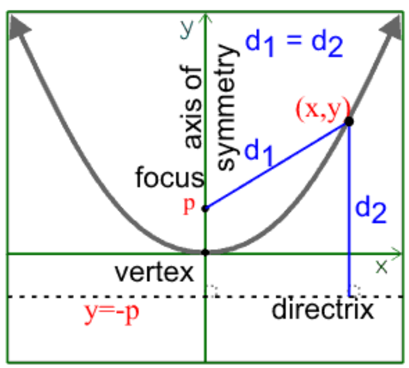
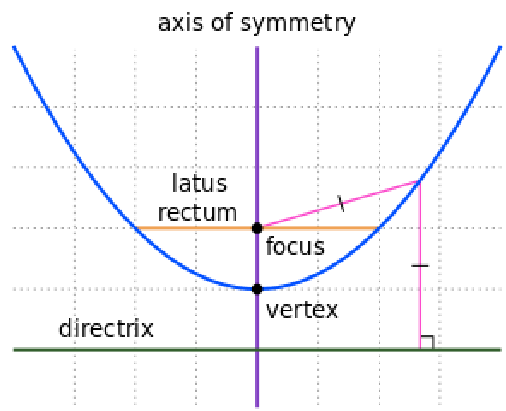
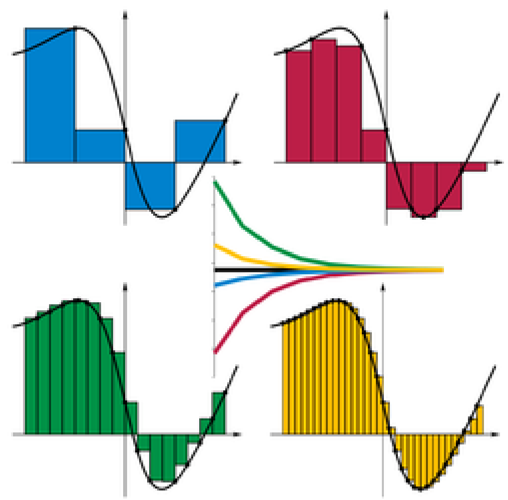
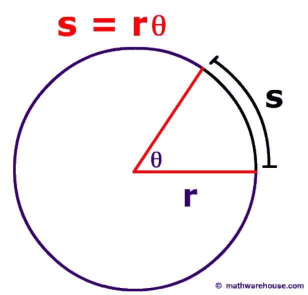
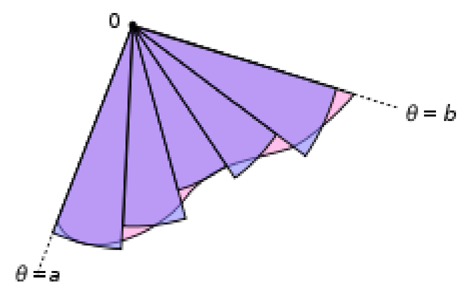
	Rectangular	Polar	Parametric
Hyperbola	$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$ *Center:* (h, k) *Vertices:* $(h \pm a, k)$ *Foci:* $(h \pm c, k)$ *Focus length, c, from center:* $c^2 = a^2 + b^2$ 	*Vertical Axis of Symmetry:* $r = \frac{a\,(e^2-1)}{1 + e\cos\theta}$ $for\ e > 1$ *Eccentricity:* $e > 1$ *where* $e = \frac{c}{a} = \frac{\sqrt{a^2+b^2}}{a} = \sec\theta > 1$ *relative to center (h, k)*  *p = semi-latus rectum* *or the line segment running from the focus to the curve in the directions* $\theta = \pm\frac{\pi}{2}$ ***Interesting Note:*** *The <u>difference</u> between the distances from each focus to a point on the curve is constant.* $\lvert d_1 - d_2 \rvert = k$	*Left-Right Opening Hyperbola:* $x(t) = a\sec(t) + h$ $y(t) = b\tan(t) + k$ $[t_{min}, t_{max}] = [-c, c]$ *Vertex: (h, k)* *Up-Down Opening Hyperbola:* $x(t) = a\tan(t) + h$ $y(t) = b\sec(t) + k$ $[t_{min}, t_{max}] = [-c, c]$ *Vertex: (h, k)* *General Form:* $x(t) = At^2 + Bt + C$ $y(t) = Dt^2 + Et + F$ *where A and D have different signs*

	Rectangular	Polar	Parametric
Parabola	*Vertical Axis of Symmetry:* $x^2 = 4py$ $(x-h)^2 = 4p(y-k)$ *Vertex:* (h,k) *Focus:* $(h, k+p)$ *Directrix:* $y = k - p$ *Horizontal Axis of Symmetry:* $y^2 = 4px$ $(y-k)^2 = 4p(x-h)$ *Vertex:* (h,k) *Focus:* $(h+p, k)$ *Directrix:* $x = h - p$ 	*Vertical Axis of Symmetry:* $r = \dfrac{2d}{1 + e\cos\theta}$ *Eccentricity:* $e = 1$ *and* $d = 2p$ *Trigonometric Form:* $y = x^2$ $r\sin\theta = r^2\cos^2\theta$ $r = \dfrac{\sin\theta}{\cos^2\theta} = \tan\theta\sec\theta$  ***Interesting Note:*** *The distances from a point on the curve to the focus is the <u>same</u> as to the directrix.*	*Vertical Axis of Symmetry:* $x(t) = 2pt + h$ $y(t) = pt^2 + k$ *(opens upwards)* $y(t) = -pt^2 - k$ *(opens downwards)* $[t_{min}, t_{max}] = [-c, c]$ *Vertex:* (h,k) *Horizontal Axis of Symmetry:* $y(t) = 2pt + k$ $x(t) = pt^2 + h$ *(opens to the right)* $x(t) = -pt^2 - h$ *(opens to the left)* $[t_{min}, t_{max}] = [-c, c]$ *Vertex:* (h,k) *Projectile Motion:* $x(t) = x_0 + v_x t + \left(\frac{1}{2}\right) a_x t^2$ $y(t) = y_0 + v_{y0}t - 16t^2$ *feet* $y(t) = y_0 + v_{y0}t - 4.9t^2$ *meters* $v_x = v\cos\theta$ $v_y = v\sin\theta$ *General Form:* $x = At^2 + Bt + C$ $y = Dt^2 + Et + F$ *where A and D have the same sign*

	Rectangular	Polar	Parametric
1st Derivative	$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ $f'(x) = \frac{dy}{dx} = y' = D_x$	$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$ *Hint: Use Product Rule for* $y = r\sin\theta$ $x = r\cos\theta$	$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, $\quad provided\ \frac{dx}{dt} \neq 0$
2nd Derivative	$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y'' = D_{xx}$	$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$	$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}}$
Integral	*Fundamental Theorem of Calculus:* $F(x) = \int_a^b f(x)\,dx \; = F(b) - F(a)$		*Riemann Sum:* $S = \sum_{i-1}^{n} f(y_i)(x_i - x_{i-1})$ *Left Sum:* $S = \left(\frac{1}{n}\right)\left[f(a) + f\left(a+\frac{1}{n}\right) + f\left(a+\frac{2}{n}\right) + \cdots + f(b - \frac{1}{n})\right]$ *Middle Sum:* $S = \left(\frac{1}{n}\right)\left[f\left(a+\frac{1}{2n}\right) + f\left(a+\frac{3}{2n}\right) + \cdots + f(b - \frac{1}{2n})\right]$ *Right Sum:* $S = \left(\frac{1}{n}\right)\left[f\left(a+\frac{1}{n}\right) + f\left(a+\frac{2}{n}\right) + \cdots + f(b)\right]$

	Rectangular	Polar	Parametric
Inverse Functions	$f(f^{-1}(x)) = f^{-1}(f(x)) = x$ *Inverse Function Theorem:* $f^{-1}(f'(a)) = \frac{1}{f'(a)}$	$if\ y = sin\,\theta$ $then\ \theta = sin^{-1}y$ $or\ \theta = \arcsin y$ $if\ y = cos\,\theta$ $then\ \theta = cos^{-1}y$ $or\ \theta = \arccos y$ $if\ y = tan\,\theta$ $then\ \theta = tan^{-1}y$ $or\ \theta = arctan\,y$ $if\ y = csc\,\theta$ $then\ \theta = csc^{-1}y$ $or\ \theta = arccsc\,y$ $if\ y = sec\,\theta$ $then\ \theta = sec^{-1}y$ $or\ \theta = arcsec\,y$ $if\ y = cot\,\theta$ $then\ \theta = cot^{-1}y$ $or\ \theta = arccot\,y$	
Arc Length	$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$ *Proof:* $\Delta s = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ $ds = \sqrt{dx^2 + dy^2}$ $ds = \sqrt{dx^2 + dy^2\left(\frac{dx^2}{dx^2}\right)}$ $ds = \sqrt{dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2}$ $ds = \sqrt{dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$ $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ $L = \int ds$	$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ *Circle:* $L = s = r\theta$ *Proof:* $L = (fraction\ of\ circumference)\cdot$ $\pi\cdot(diameter)$ $L = \left(\frac{\theta}{2\pi}\right)\pi\,(2r) = r\theta$ 	$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$ *Proof:* $ds = \sqrt{dx^2 + dy^2}$ $ds = \sqrt{dx^2\left(\frac{dt^2}{dt^2}\right) + dy^2\left(\frac{dt^2}{dt^2}\right)}$ $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 dt^2}$ $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ $L = \int ds$
Perimeter	*Square:* $P = 4s$ *Rectangle:* $P = 2l + 2w$ *Triangle:* $P = a + b + c$ *Circle:* $C = \pi d = 2\pi r$ *Ellipse:* $C \approx \pi(a + b)$	*Ellipse:* $C \approx 2\pi\sqrt{\frac{a^2+b^2}{2}}$ $C \approx \pi\left[3(a + b) - \sqrt{(3a + b)(a + 3b)}\right]$ $C \approx \pi\,(a + b)\left(1 + \frac{3h}{10 + \sqrt{4 - 3h}}\right)$	*Ellipse:* $C = 4a\int_0^{\frac{\pi}{2}}\sqrt{1 - k^2 sin^2\theta}\, d\theta$ $h = \frac{(a - b)^2}{(a + b)^2}$ & $k^2 = \left(1 - \frac{b^2}{a^2}\right)$

	Rectangular	Polar	Parametric
Area	Square: $A = s^2$ Rectangle: $A = lw$ Rhombus: $A = \frac{1}{2} ab$ Parallelogram: $A = Bh$ Trapezoid: $A = \frac{(B_1 + B_2)}{2} h$ Kite: $A = \frac{d_1 d_2}{2}$ Triangle: $A = \frac{1}{2} Bh$ Triangle: $A = \frac{1}{2} ab \sin(C)$ Triangle using Heron's Formula: $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$ Equilateral Triangle: $A = \frac{1}{4}\sqrt{3}s^2$ Frustum: $A = \frac{1}{3}\left(\frac{B_1+B_2}{2}\right)h$ Circle: $A = \pi r^2$ Circular Sector: $A = \frac{1}{2} r^2\theta$ Ellipse: $A = \pi ab$	$A = \int_\alpha^\beta \frac{1}{2}[f(\theta)]^2 d\theta$ where $r = f(\theta)$ Proof: Area of a sector: $A = \int s\,dr = \int r\,\Delta\theta\,dr = \frac{1}{2}r^2\,\Delta\theta$ where arc length $s = r\,\Delta\theta$ 	$A = \int_\alpha^\beta g(t)\,f'(t)\,dt$ where $f(t) = x$ and $g(t) = y$ or $x(t) = f(t)$ and $y(t) = g(t)$ Simplified: $A = \int_\alpha^\beta y(t)\frac{dx(t)}{dt}\,dt$ Proof: $\int_a^b f(x)\,dx$ $y = f(x) = g(t)$ $dx = \frac{df(t)}{dt}dt = f'(t)\,dt$
Lateral Surface Area	Cylinder: $SA = 2\pi rh$ Cone: $SA = \pi rl$ $SA = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$	For rotation about the x-axis: $SA = \int 2\pi y\ ds$ For rotation about the y-axis: $SA = \int 2\pi x\ ds$ $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$ $r = f(\theta), \quad \alpha \le \theta \le \beta$	For rotation about the x-axis: $SA = \int 2\pi y\ ds$ For rotation about the y-axis: $SA = \int 2\pi x\ ds$ $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ if $x = f(t), y = g(t), \alpha \le t \le \beta$

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<tr><th></th><th>Rectangular</th><th>Polar</th><th>Parametric</th></tr>
<tr><td>Total Surface Area</td><td colspan="3">Cube: $SA = 6s^2$
Rectangular Box: $SA = 2lw + 2wh + 2hl$
Regular Tetrahedron: $SA = 2bh$
Cylinder: $SA = 2\pi r\,(r + h)$
Cone: $SA = \pi r^2 + \pi rl = \pi r\,(r + l)$
Sphere: $SA = 4\pi r^2$
Ellipsoid: $SA \approx 4\pi\left(\frac{a^pb^p+a^pc^p+b^pc^p}{3}\right)^{1/p}$
$Where\ p \approx 1.6075,\ \ |Relative\ Error| \le 1.061\%$
(Knud Thomsen's Formula)</td></tr>
<tr><td>Surface of Revolution</td><td>For revolution about the x-axis:
$$A = 2\pi\int_a^b f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
For revolution about the y-axis:
$$A = 2\pi\int_a^b x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$</td><td>For revolution about the x-axis:
$$A = 2\pi r\int_\alpha^\beta \cos\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$
For revolution about the y-axis:
$$A = 2\pi r\int_\alpha^\beta \sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$</td><td>For revolution about the x-axis:
$$A = 2\pi\int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$
For revolution about the y-axis:
$$A = 2\pi\int_a^b x(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$</td></tr>
<tr><td>Volume</td><td colspan="3">Cube: $V = s^3$
Rectangular Prism: $V = lwh$
Cylinder: $V = \pi r^2 h$
Triangular Prism: $V = Bh$
Tetrahedron: $V = \frac{1}{3}Bh$
Pyramid: $V = \frac{1}{3}Bh = \frac{1}{3}lwh$
Cone: $V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h$
Sphere: $V = \frac{4}{3}\pi r^3$
Ellipsoid: $V = \frac{4}{3}\pi abc$</td></tr>
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<tr><td rowspan="3">Volume of Revolution</td><td>

<u>Disk Method</u>

$$V = \int_a^b (area\ of\ circle)\ d(thickness)$$

Rotation about the x-axis:

$$V = \int_a^b \pi\,[f(x)]^2 dx$$

Rotation about the y-axis:

$$V = \int_c^d \pi x^2 dy$$

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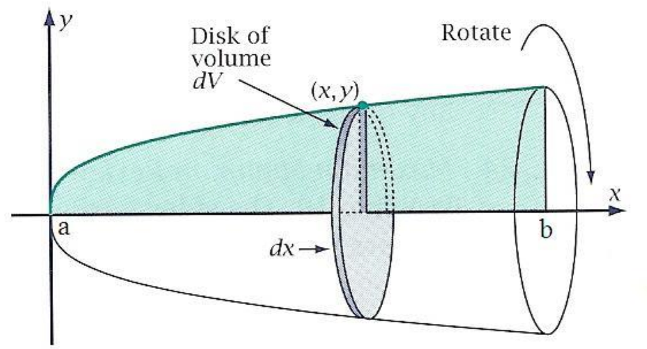

</td></tr>
<tr><td>

<u>Washer Method</u>

Rotation about the x-axis:

$$V = \int_a^b \pi\,\{[f(x)]^2 - [g(x)]^2\}\,dx$$

</td><td colspan="2">

$$V = V_{Outer\ Disk} - V_{Inner\ Disk}$$

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<tr><td>

<u>Shell Method</u>

$$V = \int_a^b (circumference)\ (hight)\ dx$$

Rotation about the y-axis:

$$V = \int_a^b 2\pi x\, f(x)\, dx$$

Rotation about the x-axis:

$$V = \int_c^d 2\pi y\, g(y)\, dy$$

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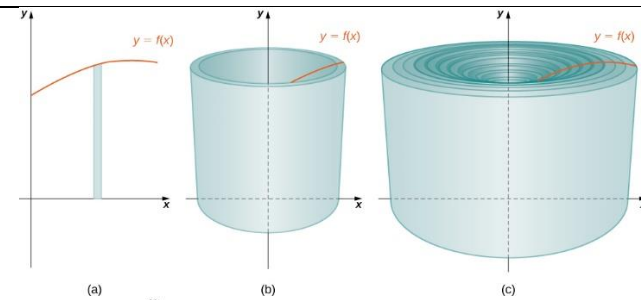

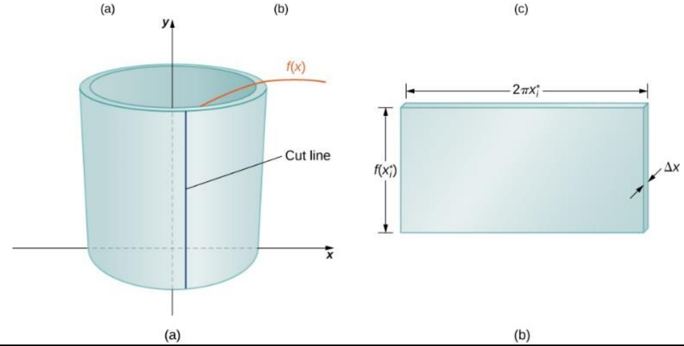

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	Rectangular	Polar	Parametric

