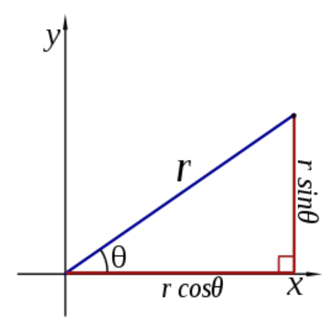
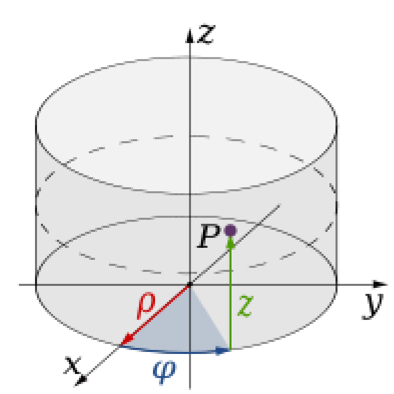
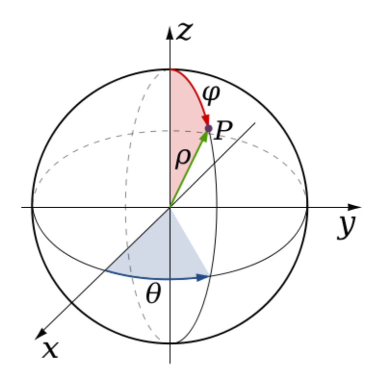
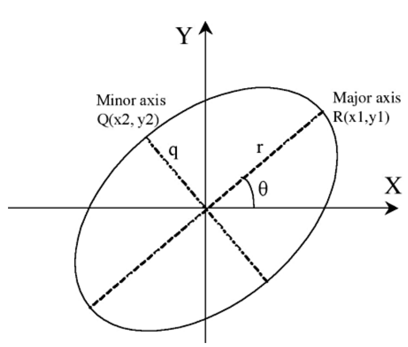
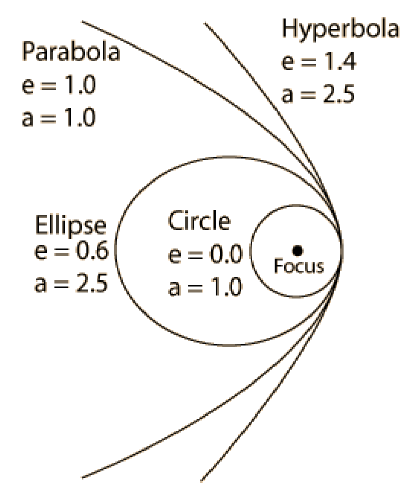
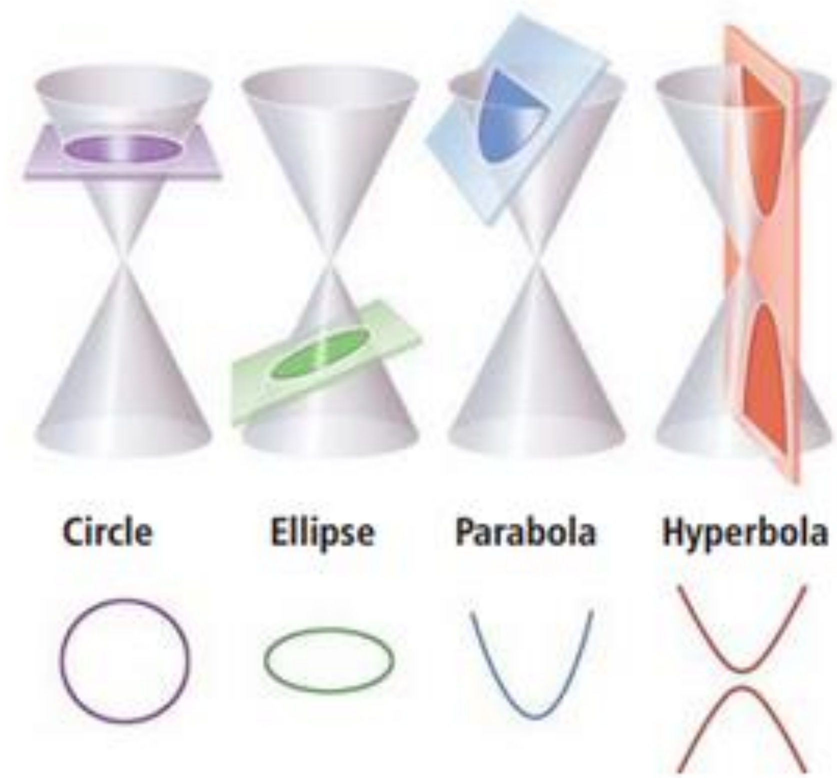
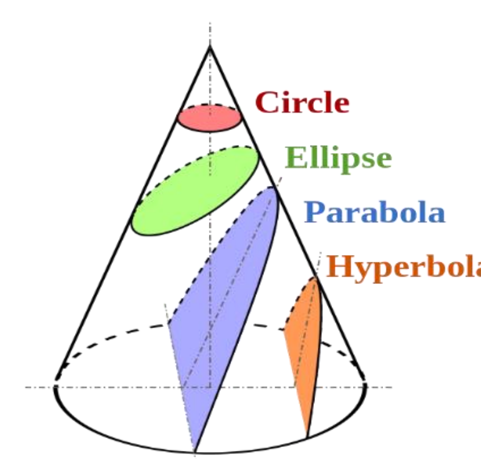
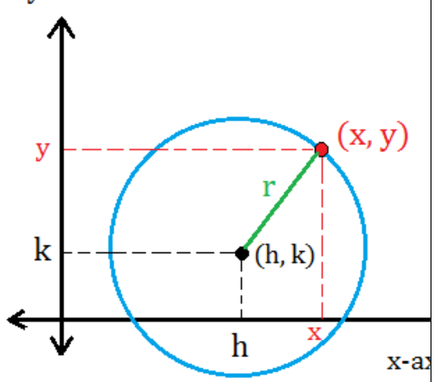
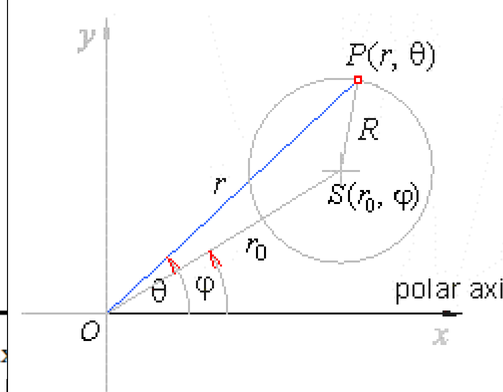
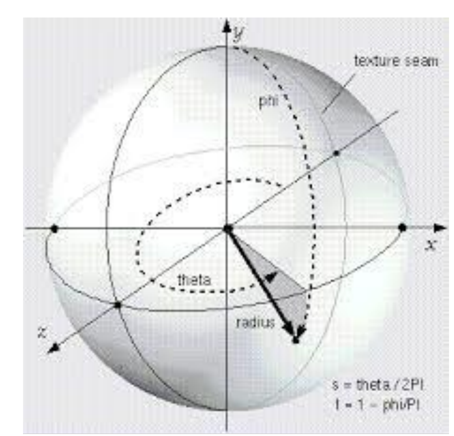
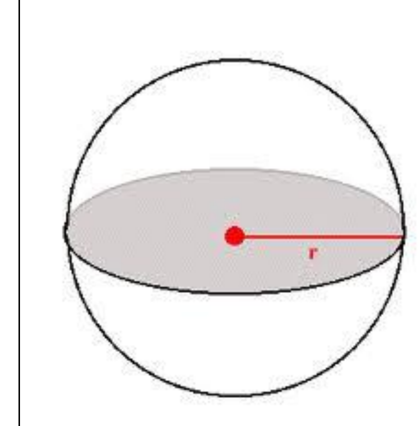
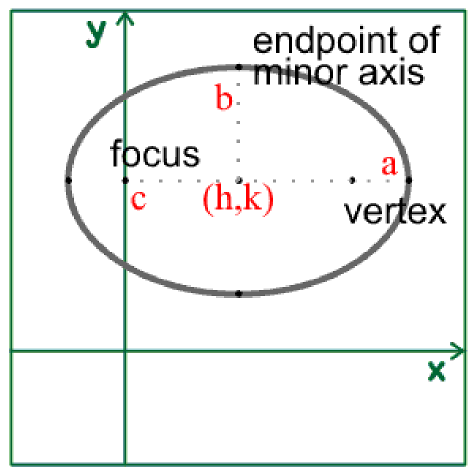
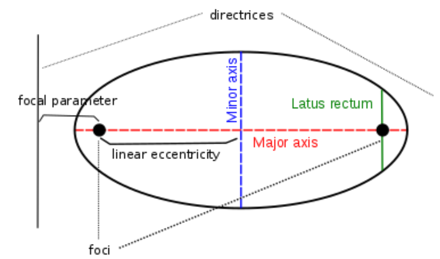
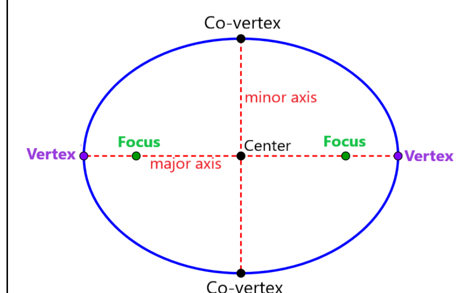
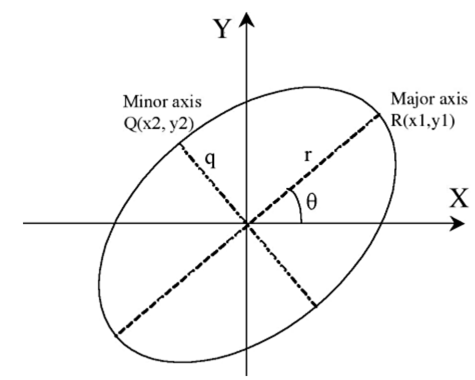
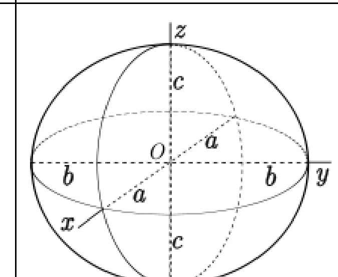
Harold's Calculus 3
Multi-Coordinate System
Cheat Sheet

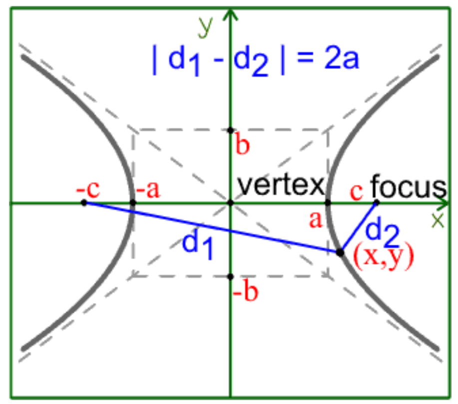
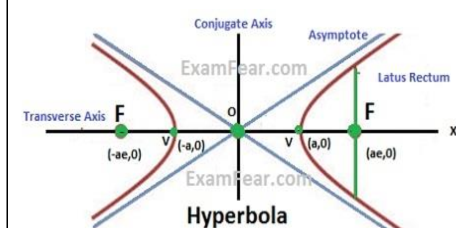
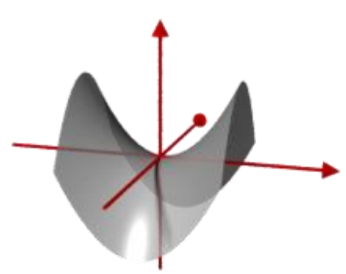
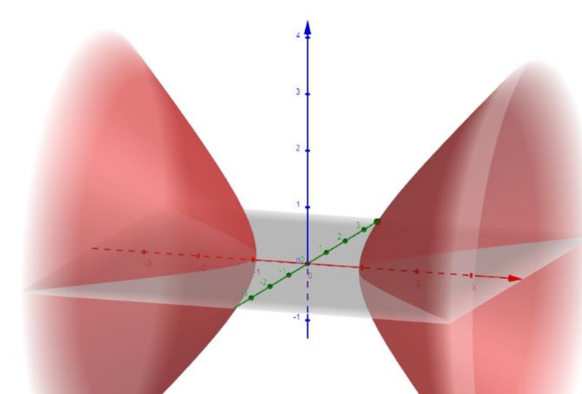
29 November 2022

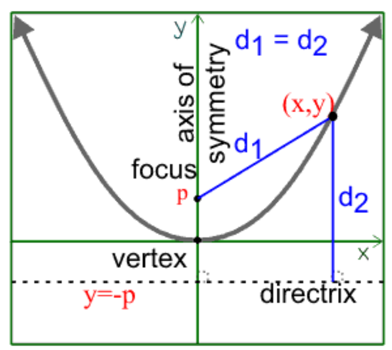
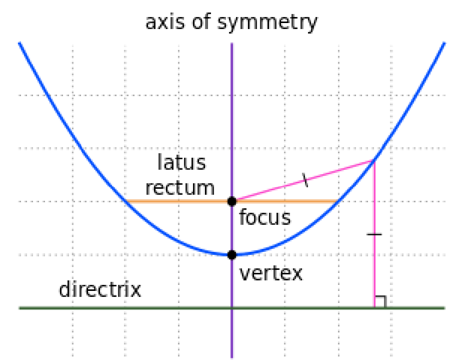
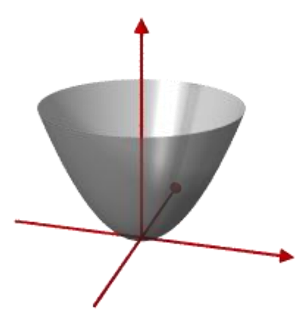
	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Point	2D $f(x) = y$ (x, y) (a, b) 3D $f(x, y) = z$ (x, y, z) 4D $f(x, y, z) = w$ (x, y, z, w)	(r, θ) or $r \angle \theta$ *Polar → Rect.* $x = r\cos\theta$ $y = r\sin\theta$ $z = z$ $\tan\theta = \frac{y}{x}$ *Rect. → Polar* $r^2 = x^2 + y^2$ $r = \pm\sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$	(ρ, θ, ϕ) $x = \rho \sin\phi \cos\theta$ $y = \rho \sin\phi \sin\theta$ $z = \rho \cos\phi$ $\rho^2 = r^2 + z^2$ $\rho^2 = x^2 + y^2 + z^2$ $\tan\theta = \left(\frac{y}{x}\right)$ $\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$ $\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$	*Point (a,b) in Rectangular :* $x(t) = a$ $y(t) = b$ $< a, b >$ $t = 3^{rd}$ *variable, usually time, with 1 degree of freedom (df)*	$\vec{r} = \langle x_0, y_0, z_0 \rangle$ $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$	$[a] \quad [x] = [b]$
Line	*Slope-Intercept Form:* $y = mx + b$ *Point-Slope Form:* $y - y_0 = m(x - x_0)$ *General Form:* $Ax + By + C = 0$ *where A and B* $\neq 0$ *Calculus Form:* $f(x) = f'(a)\,x + f(0)$ *where* $m = f'(a)$ *3-D:* $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$	  	 	$< x, y > = < x_0, y_0 > + t < a, b >$ $< x, y > = < x_0 + at, y_0 + bt >$ *where* $< a, b > = < x_2 - x_1,\ y_2 - y_1 >$ $x(t) = x_0 + ta$ $y(t) = y_0 + tb$ $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{b}{a}$	$\vec{r} = \vec{r_0} + t\,\vec{v}$ $= \langle x_0, y_0, z_0 \rangle$ $+ t\,\langle a, b, c \rangle$	$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [c]$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$

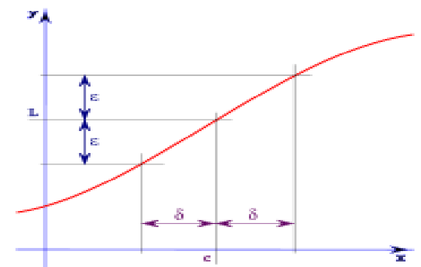
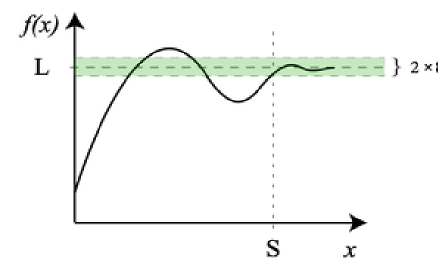
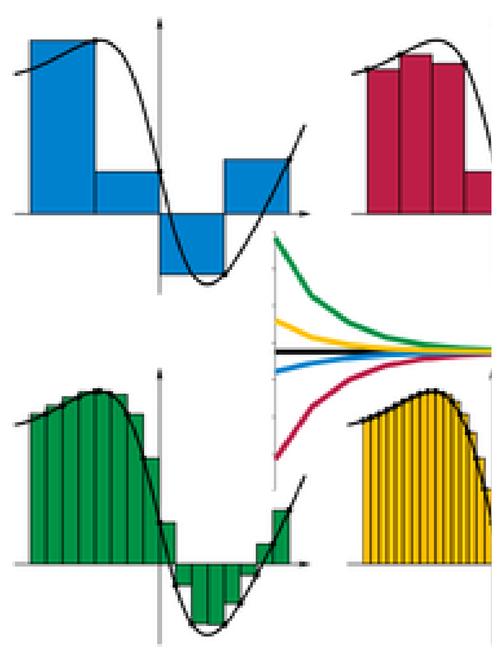
	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Plane	$a(x - x_0)$ $+ b(y - y_0)$ $+ c(z - z_0) = 0$ $ax + by + cz = d$ where $d = ax_0 + by_0 + cz_0$ $f(x, y) = Ax + By + C$	$(r, \theta, constant)$ $(0 \le r < \infty)$ $(0 \le \theta < 2\pi)$ *where r and θ take on all values in their domains*	$(\rho, \theta, constant)$ $(0 \le \rho < \infty)$ $(0 \le \theta < 2\pi)$ *where ρ and θ take on all values in their domains*	$\boldsymbol{r} = \boldsymbol{r}_0 + s\boldsymbol{v} + t\boldsymbol{w}$ *where:* • *s and t range over all real numbers* • ***v** and **w** are given vectors defining the plane* • *$\boldsymbol{r_0}$ is the vector representing the position of an arbitrary (but fixed) point on the plane*	$\boldsymbol{n} \bullet (\boldsymbol{r} - \boldsymbol{r}_0) = 0$	
Conics	*General Equation for All Conics:* $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ *where* *Line*: $A = B = C = 0$ *Circle*: $A = C$ *and* $B = 0$ *Ellipse*: $AC > 0$ *or* $B^2 - 4AC < 0$ *Parabola*: $AC = 0$ *or* $B^2 - 4AC = 0$ *Hyperbola*: $AC < 0$ *or* $B^2 - 4AC > 0$ *Note: If* $A + C = 0$, *square hyperbola* *Rotation:* *If B ≠ 0, then* [rotate]() *coordinate system:* $\cot 2\theta = \frac{A - C}{B}$ $x = x' \cos\theta - y' \sin\theta$ $y = y' \cos\theta + x' \sin\theta$ *New = (x', y'), Old = (x, y) rotates through angle θ from x-axis* 	*General Equation for All Conics:* *Vertical Axis of Symmetry:* $r = \frac{p}{1 - e\cos\theta}$ *Horizontal Axis of Symmetry:* $r = \frac{p}{1 - e\sin\theta}$ $where\ p = \begin{cases} a(1 - e^2) \\ 2d \\ a(e^2 - 1) \end{cases} for \begin{cases} 0 \le e < 1 \\ e = 1 \\ e > 1 \end{cases}$ *p = semi-latus rectum or the line segment running from the focus to the curve in a direction parallel to the directrix* *Eccentricity:* *Circle* $e = 0$ *Ellipse* $0 < e < 1$ *Parabola* $e = 1$ *Hyperbola* $e > 1$ 	 			NA

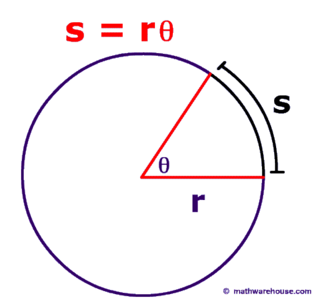
	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Circle	$x^2 + y^2 = r^2$ $(x-h)^2 + (y-k)^2 = r^2$ *General Form:* $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ *where* $A = C$ *and* $B = 0$ *Center:* (h, k) *Vertices: NA* *Focus:* (h, k) 	*Centered at Origin:* *r = a (constant)* $\theta = \theta \quad [0, 2\pi]$ *or* $[0, 360°]$ *Centered at* (r_0, ϕ)*:* $r^2 + r_0^2 - 2rr_0\cos(\theta - \phi) = R^2$ *Hint: Law of Cosines* *or* $r = r_0\cos(\theta-\phi) + \sqrt{a^2 - r_0^2\sin^2(\theta-\phi)}$ 	$\rho =$ *constant* $\theta = \theta\ [0, 2\pi]$ $\phi =$ *constant* $= 0$	$x(t) = r\cos(t) + h$ $y(t) = r\sin(t) + k$ $[t_{min}, t_{max}] = [0, 2\pi)$ *Center:* (h, k) *Focus:* (h, k)	NA	NA
Sphere	$x^2 + y^2 + z^2 = r^2$ $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$ *Focus and center:* (h, k, l) *General Form:* $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$ *where* $A = B = C > 0$ *Cylindrical to Rectangular:* $x = r\cos(\theta)$ $y = r\sin(\theta)$ $z = z$ Spherical to Rectangular: $x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	*Rectangular to Cylindrical:* $r = \sqrt{x^2 + y^2}$ *Spherical to Cylindrical:* $\rho = r\sin(\theta)$ $\phi = \phi$ $z = r\cos(\theta)$	$\rho =$ *constant* $\theta = \theta\ [0, 2\pi]$ $\phi = \phi\ [0, 2\pi]$ *Rectangular to Spherical:* $r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \arccos\left(\frac{z}{r}\right)$ $\phi = \arctan\left(\frac{y}{x}\right)$ *Cylindrical to Spherical:* $r = \sqrt{\rho^2 + z^2}$ $\theta = \arctan\left(\frac{\rho}{z}\right) = \arccos\left(\frac{z}{r}\right)$ $\phi = \phi$		Rectangular: $\boldsymbol{r} \equiv \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ Cylindrical: $\boldsymbol{r} \equiv \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ z \end{bmatrix}$ Spherical: $\boldsymbol{r} \equiv \begin{bmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{bmatrix}$	

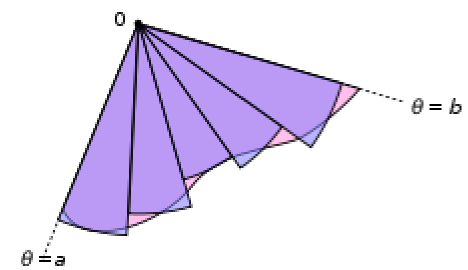
	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Ellipse	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ *General Form:* $Ax^2+Bxy+Cy^2+Dx+Ey$ $+F=0$ *where* $B^2-4AC<0$ *or* $AC>0$ *Center:* (h,k) *Vertices:* $(h\pm a,k)$ *Co-Vertices:* $(h,k\pm b)$ *Foci:* $(h\pm c,k)$ *Focus length, c, from center:* $c^2=\ a^2-b^2$ *Eccentricity:* $e=\frac{c}{a}=\frac{\sqrt{a^2-\ b^2}}{a}$ *If B ≠ 0, then rotate coordinate system:* $\cot 2\theta=\frac{A-C}{B}$ $x=x'\cos\theta-y'\sin\theta$ $y=y'\cos\theta+x'\sin\theta$ *New = (x', y'), Old = (x, y) rotates through angle θ from x-axis*	*Vertical Axis of Symmetry:* $r=\frac{a\,(1-e^2)}{1\pm e\cos\theta}$ *Horizontal Axis of Symmetry:* $r=\frac{a\,(1-e^2)}{1\pm e\sin\theta}$ *Eccentricity*: $0<e<1$ $r(\theta)=\frac{a\,b}{\sqrt{(b\cos\theta)^2+(a\sin\theta)^2}}$ *relative to center* (h,k) 	  ***Interesting Note:*** *The sum of the distances from each focus to a point on the curve is constant.* $\|d_1+d_2\|=k$	$x(t)=a\cos(t)\ +h$ $y(t)=b\sin(t)+\ k$ $[t_{min},\ t_{max}]=[0,2\pi]$ *Center:* (h,k) *Rotated Ellipse:* $x(t)=a\cos t\cos\theta-b\sin t\sin\theta+h$ $y(t)=a\cos t\sin\theta+b\sin t\cos\theta+k$ *θ = the angle between the x-axis and the major axis of the ellipse* 		
Ellipsoid	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}+\frac{(z-l)^2}{c^2}$ $=1$	$\frac{r^2\cos^2\theta}{a^2}+\frac{r^2\sin^2\theta}{b^2}+\frac{z^2}{c^2}=1$	$\frac{r^2\cos^2\theta\,\sin^2\phi}{a^2}$ $+\frac{r^2\sin^2\theta\,\sin^2\phi}{b^2}$ $+\frac{r^2\cos^2\phi}{c^2}=1$	$x(t,u)=a\cos(t)\cos(u)+h$ $y(t,u)=b\cos(t)\sin(u)+\ k$ $z(t,u)=c\sin(t)+\ l$ $[t_{min},\ t_{max}]=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ $[u_{min},\ u_{max}]=[-\pi,\pi]$ *Center*: (h,k,l)		$(\boldsymbol{x}-\boldsymbol{v})^T A^{-1}(\boldsymbol{x}-\boldsymbol{v})=1$ *Centered at vector* $\boldsymbol{v}$

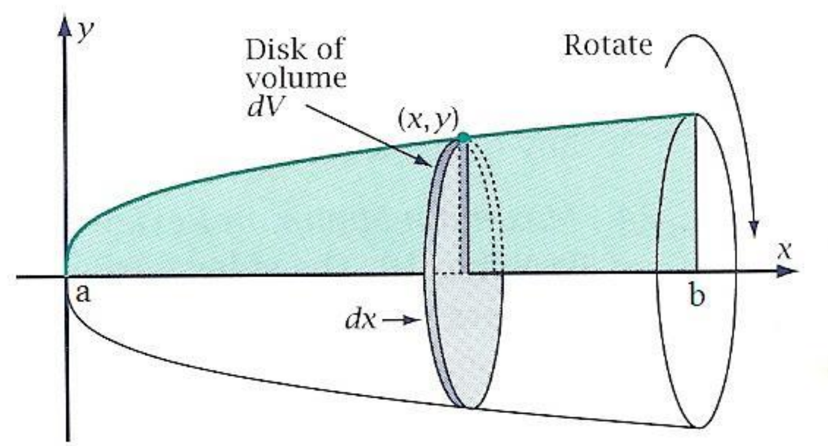
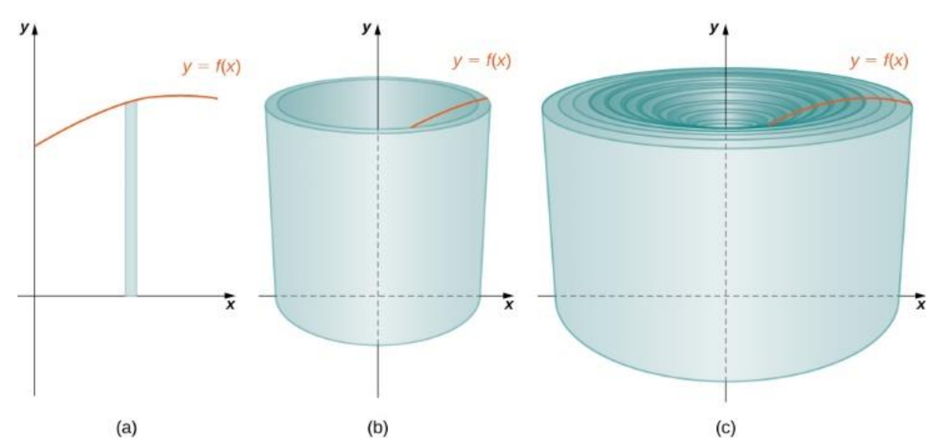
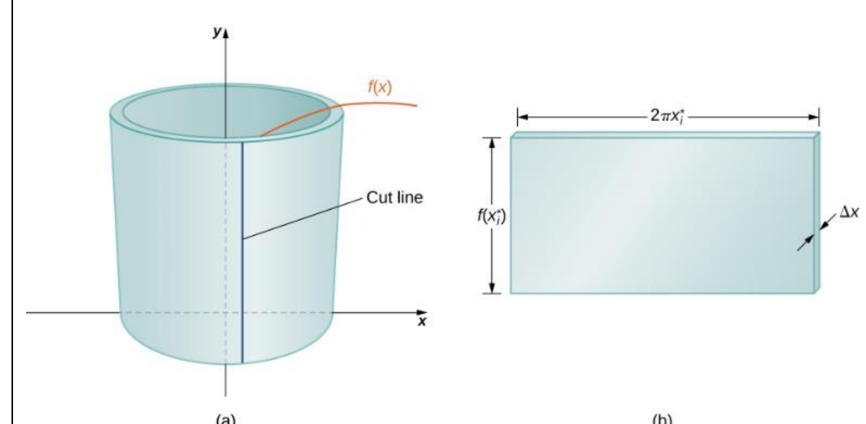
	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ *General Form:* $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ *where* $B^2 - 4AC > 0$ *or* $AC < 0$ *If* $A + C = 0$*, square hyperbola* *Center:* (h, k) *Vertices:* $(h \pm a, k)$ *Foci:* $(h \pm c, k)$ *Focus length, c, from center:* $c^2 = a^2 + b^2$ *Eccentricity:* $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} = \sec\theta$ *If B ≠ 0, then rotate coordinate system:* $\cot 2\theta = \frac{A - C}{B}$ $x = x'\cos\theta - y'\sin\theta$ $y = y'\cos\theta + x'\sin\theta$ *New = (x', y'), Old = (x, y) rotates through angle θ from x-axis*	 ***Interesting Note:*** *The difference between the distances from each focus to a point on the curve is constant.* $\lvert d_1 - d_2 \rvert = k$	*Vertical Axis of Symmetry:* $r = \frac{a\,(e^2 - 1)}{1 \pm e\cos\theta}$ *Horizontal Axis of Symmetry:* $r = \frac{a\,(e^2 - 1)}{1 \pm e\sin\theta}$ *Eccentricity:* $e > 1$ *where* $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} = \sec\theta > 1$ *relative to center (h, k)* $-\cos^{-1}\left(-\frac{1}{e}\right) < \theta < \cos^{-1}\left(-\frac{1}{e}\right)$  *p = semi-latus rectum or the line segment running from the focus to the curve in the directions* $\theta = \pm\frac{\pi}{2}$	*Left-Right Opening Hyperbola:* $x(t) = a\sec(t) + h$ $y(t) = b\tan(t) + k$ $[t_{min}, t_{max}] = [-c, c]$ *Vertex: (h, k)* *Alternate Form:* $x(t) = \pm a\cosh(t) + h$ $y(t) = b\sinh(t) + k$ *Up-Down Opening Hyperbola:* $x(t) = a\tan(t) + h$ $y(t) = b\sec(t) + k$ $[t_{min}, t_{max}] = [-c, c]$ *Vertex: (h, k)* *Alternate Form:* $x(t) = a\sinh(t) + h$ $y(t) = \pm b\cosh(t) + k$ *General Form:* $x(t) = At^2 + Bt + C$ $y(t) = Dt^2 + Et + F$ *where A and D have different signs*		
Hyperboloid	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-l)^2}{c^2} = 1$ $-\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$					

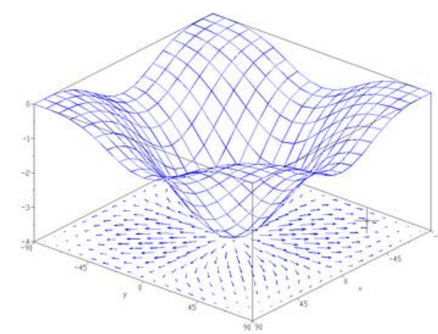
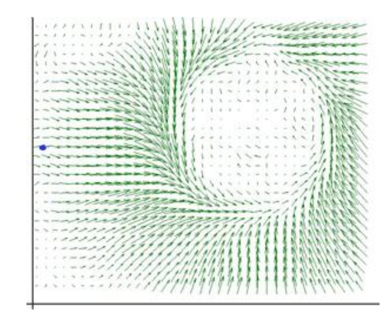
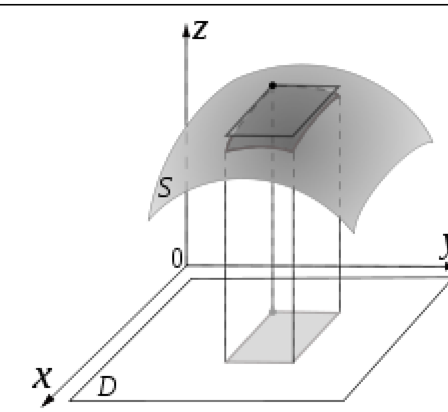
	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Parabola	*Vertical Axis of Symmetry:* $x^2 = 4py$ $(x-h)^2 = 4p(y-k)$ *Vertex:* (h,k) *Focus:* $(h,k+p)$ *Directrix:* $y = k - p$ *Horizontal Axis of Symmetry:* $y^2 = 4px$ $(y-k)^2 = 4p(x-h)$ *Vertex:* (h,k) *Focus:* $(h+p,k)$ *Directrix:* $x = h - p$ *General Form:* $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ *where* $B^2 - 4AC = 0$ *or* $AC = 0$ *If B ≠ 0, then rotate coordinate system:* $\cot 2\theta = \frac{A-C}{B}$ $x = x' \cos\theta - y' \sin\theta$ $y = y' \cos\theta + x' \sin\theta$ *New = (x', y'), Old = (x, y) rotates through angle θ from x-axis*	*Vertical Axis of Symmetry:* $r = \frac{ed}{1 \pm e\cos\theta}$ *Horizontal Axis of Symmetry:* $r = \frac{ed}{1 \pm e\sin\theta}$ *Eccentricity:* $e = 1$ *and* $d = 2p$  	 ***Interesting Note:*** *The distances from a point on the curve to the focus is the* <u>*same*</u> *as to the directrix.*	*Vertical Axis of Symmetry:* $x(t) = 2pt + h$ $y(t) = pt^2 + k$ *(opens upwards)* $y(t) = -pt^2 - k$ *(opens downwards)* $[t_{min}, t_{max}] = [-c, c]$ *Vertex:* (h,k) *Horizontal Axis of Symmetry:* $y(t) = 2pt + k$ $x(t) = pt^2 + h$ *(opens to the right)* $x(t) = -pt^2 - h$ *(opens to the left)* $[t_{min}, t_{max}] = [-c, c]$ *Vertex:* (h,k) *Projectile Motion:* $x(t) = x_0 + v_x t + \left(\frac{1}{2}\right) a_x t^2$ $y(t) = y_0 + v_y t - 16t^2$ *feet* $y(t) = y_0 + v_y t - 4.9t^2$ *meters* $v_x = v\cos\theta$ $v_y = v\sin\theta$ *General Form:* $x = At^2 + Bt + C$ $y = Lt^2 + Mt + N$ *where A and L have the same sign*		
Paraboloid	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{(z-l)^2}{c^2}$					

	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Limit	$\lim_{x\to c} f(x) = L$					
1st Derivative	$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ $f'(x) = \frac{dy}{dx} = y' = D_x$	$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$ *Hint: Use Product Rule for* $y = r\sin\theta$ $x = r\cos\theta$		$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, *provided* $\frac{dx}{dt} \neq 0$	$\frac{d}{dt}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{r}}\,'$ *Unit tangent vector* $\vec{T}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{\lVert\vec{\boldsymbol{r}}'(t)\rVert}$ *where* $\vec{\boldsymbol{r}}'(t) \neq \vec{0}$	
2nd Derivative	$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y'' = D_{xx}$	$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$		$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}}$	*Unit normal vector* $\vec{N}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{\lVert\vec{\boldsymbol{T}}'(t)\rVert}$ *where* $\vec{\boldsymbol{T}}'(t) \neq \vec{0}$	
Integral	*Fundamental Theorem of Calculus:* $F(x) = \int_a^b f(x)\,dx = F(b) - F(a)$			*Riemann Sum:* $S = \sum_{i-1}^{n} f(y_i)(x_i - x_{i-1})$ *Left Sum:* $S = \left(\frac{1}{n}\right)\left[f(a) + f\left(a+\frac{1}{n}\right) + f\left(a+\frac{2}{n}\right) + \cdots + f(b-\frac{1}{n})\right]$ *Middle Sum:* $S = \left(\frac{1}{n}\right)\left[f\left(a+\frac{1}{2n}\right) + f\left(a+\frac{3}{2n}\right) + \cdots + f(b-\frac{1}{2n})\right]$ *Right Sum:* $S = \left(\frac{1}{n}\right)\left[f\left(a+\frac{1}{n}\right) + f\left(a+\frac{2}{n}\right) + \cdots + f(b)\right]$	$\int_a^b \vec{\boldsymbol{r}}(t)\,dt = \langle\int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt\rangle$	
Double Integral	$\int_a^b \int_{c(y)}^{d(y)} f(x,y)\,dx\,dy$	*Same as rectangular, but* $f(x,y) \longrightarrow f(\rho\cos\phi, \rho\sin\phi)$				

	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Triple Integral	$\int_a^b \int_{c(z)}^{d(z)} \int_{e(y,z)}^{g(y,z)} f(x,y,z)\, dx\, dy\, dz$	*Same as rectangular, but* $f(x,y,z) \longrightarrow f(\rho\cos\phi, \rho\sin\phi, z)$	*Same as rectangular, but* $f(x,y,z) \longrightarrow f(\rho\cos\theta\sin\phi,$ $\rho\sin\theta\sin\phi, \rho\cos\phi)$	*NA*	*NA*	*NA*
Inverse Functions	*If* $f(x) = y$, *then* $f^{-1}(y) = x$ *Inverse Function Theorem:* $f^{-1}(f'(a)) = \frac{1}{f'(a)}$	*if* $y = \sin\theta$ *then* $\theta = \sin^{-1} y$ *if* $y = \cos\theta$ *then* $\theta = \cos^{-1} y$ *if* $y = \tan\theta$ *then* $\theta = \tan^{-1} y$ *if* $y = \csc\theta$ *then* $\theta = \csc^{-1} y$ *if* $y = \sec\theta$ *then* $\theta = \sec^{-1} y$ *if* $y = \cot\theta$ *then* $\theta = \cot^{-1} y$	*or* $\theta = \arcsin y$ *or* $\theta = \arccos y$ *or* $\theta = \arctan y$ *or* $\theta = \text{arccsc}\, y$ *or* $\theta = \text{arcsec}\, y$ *or* $\theta = \text{arccot}\, y$	*NA*	*NA*	*NA*
Arc Length	$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$ *Proof:* $\Delta s = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ $ds = \sqrt{dx^2 + dy^2}$ $ds = \sqrt{dx^2 + dy^2 \left(\frac{dx^2}{dx^2}\right)}$ $ds = \sqrt{dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2}$ $ds = \sqrt{dx^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)}$ $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ $L = \int ds$	*Polar:* $L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ *Where* $r = f(\theta)$ *Circle:* $L = s = r\theta$ *Proof:* $L = (fraction\ of\ circumference) \cdot$ $\pi \cdot (diameter)$ $L = \left(\frac{\theta}{2\pi}\right)\pi\,(2r) = r\theta$	*C = πd = 2πr* 	*Rectangular 2D:* $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ *Rectangular 3D:* $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$ *Cylindrical:* $L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$ *Spherical:* $L =$ $\int_{t_1}^{t_2} \sqrt{\left(\frac{d\rho}{dt}\right)^2 + \rho^2 sin^2\varphi\left(\frac{d\theta}{dt}\right)^2 + \rho^2\left(\frac{d\varphi}{dt}\right)^2}$	$L = \int_a^b \Vert \vec{r}'(t) \Vert\, dt$ $s(t) = \int_0^t \Vert \vec{r}'(u) \Vert\, du$	*NA*
Curvature	$\kappa = \frac{\lvert y'' \rvert}{(1 + y'^2)^{3/2}}$	$\kappa(\theta) = \frac{\lvert r^2 + 2r'^2 - rr'' \rvert}{(r^2 + r'^2)^{3/2}}$ *for r(θ)*	*NA*	$\kappa = \frac{\sqrt{(z''y' - y''z')^2 + (x''z' - z''x')^2 + (y''x - x''y')^2}}{(x'^2 + y'^2 + z'^2)^{3/2}}$ *where f(t) = (x(t), y(t), z(t))*	$\kappa = \left\lvert \frac{d\vec{T}}{ds} \right\rvert$ $\kappa = \frac{\Vert \vec{T}'(t) \Vert}{\Vert \vec{r}'(t) \Vert}$ $\kappa = \frac{\Vert \vec{r}'(t) \times \vec{r}''(t) \Vert}{\Vert \vec{r}'(t) \Vert^3}$	*(See Wikipedia : Curvature)*
Perimeter	*Square:* $P = 4s$ *Rectangle:* $P = 2l + 2w$ *Triangle:* $P = a + b + c$ *Circle:* $C = \pi d = 2\pi r$ *Ellipse:* $C \approx \pi(a + b)$	*Ellipse:* $C \approx 2\pi\sqrt{\frac{a^2+b^2}{2}}$ C $\approx \pi\left[3(a + b) - \sqrt{(3a + b)(a + 3b)}\right]$ $C \approx \pi\,(a + b)\left(1 + \frac{3h}{10 + \sqrt{4 - 3h}}\right)$	*Ellipse:* $C = 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 sin^2\theta}\, d\theta$ $h = \frac{(a - b)^2}{(a + b)^2}$ & $k^2 = \left(1 - \frac{b^2}{a^2}\right)$	*NA*	*NA*	*NA*

	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Area	*Square:* $A = s^2$ *Rectangle:* $A = lw$ *Rhombus:* $A = ½\, ab$ *Parallelogram:* $A = Bh$ *Trapezoid:* $A = \frac{(B_1 + B_2)}{2}\, h$ *Kite:* $A = \frac{d_1 d_2}{2}$ *Triangle:* $A = ½\, Bh$ *Triangle:* $A = ½\, ab \sin(C)$ *Triangle using Heron's Formula:* $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$ *Equilateral Triangle:* $A = ¼\sqrt{3}s^2$ *Frustum:* $A = \frac{1}{3}\left(\frac{B_1+B_2}{2}\right) h$ *Circle:* $A = \pi r^2$ *Circular Sector:* $A = ½\, r^2\theta$ *Ellipse:* $A = \pi ab$	$A = \int_\alpha^\beta \frac{1}{2}[f(\theta)]^2 d\theta$ where $r = f(\theta)$ *Proof:* *Area of a sector:* $A = \int s\, dr = \int r\,\Delta\theta\, dr = \frac{1}{2}r^2\,\Delta\theta$ where arc length $s = r\,\Delta\theta$ 	NA	$A = \int_\alpha^\beta g(t)\, f'(t)\, dt$ where $f(t) = x$ and $g(t) = y$ or $x(t) = f(t)$ and $y(t) = g(t)$ *Simplified:* $A = \int_\alpha^\beta y(t)\, \frac{dx(t)}{dt}\, dt$ *Proof:* $\int_a^b f(x)\, dx$ $y = f(x) = g(t)$ $dx = \frac{df(t)}{dt} dt = f'(t)\, dt$	$A = \iint_D \left\lvert \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\rvert du\, dv$	NA
Lateral Surface Area	*Cylinder:* $SA = 2\pi rh$ *Cone:* $SA = \pi rl$ $SA = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx$	*For rotation about the x-axis:* $SA = \int 2\pi y\, ds$ *For rotation about the y-axis:* $SA = \int 2\pi x\, ds$ $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ $r = f(\theta), \quad \alpha \le \theta \le \beta$	*Sphere:* $SA = 4\pi r^2$	*For rotation about the x-axis:* $SA = \int 2\pi y\, ds$ *For rotation about the y-axis:* $SA = \int 2\pi x\, ds$ $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ $if\ x = f(t), y = g(t), \alpha \le t \le \beta$	NA	NA
Total Surface Area	*Cube:* $SA = 6s^2$ *Rectangular Box:* $SA = 2lw + 2wh + 2hl$ *Regular Tetrahedron:* $SA = 2bh$ *Cylinder:* $SA = 2\pi r\,(r + h)$ *Cone:* $SA = \pi r^2 + \pi rl = \pi r\,(r + l)$ *Sphere:* $SA = 4\pi r^2$	*Ellipsoid:* $SA \approx$ $4\pi\left(\frac{a^p b^p + a^p c^p + b^p c^p}{3}\right)^{1/p}$ *Where* $p \approx 1.6075$, $\lvert Relative\ Error \rvert \le 1.061\%$ *(Knud Thomsen's Formula)*	*Ellipsoid:* $S =$	$\int_0^{2\pi}\int_0^{\pi} \mathrm{Sin}[\theta]\sqrt{b^2 c^2 \mathrm{Sin}[\theta]^2 \mathrm{Cos}[\phi]^2 + a^2 c^2 \mathrm{Sin}[\theta]^2 \mathrm{Sin}[\phi]^2 + a^2 b^2 \mathrm{Cos}[\theta]^2}\, d\theta\, d\phi =$ $2\pi\left(c^2 + \frac{b\,c^2}{\sqrt{a^2 - c^2}}\,\mathrm{EllipticF}[\theta, m] + b\sqrt{a^2 - c^2}\,\mathrm{EllipticE}[\theta, m]\right)$ where $m = \frac{a^2\,(b^2 - c^2)}{b^2\,(a^2 - c^2)}$; $\theta = \mathrm{ArcSin}\left[\sqrt{1 - \frac{c^2}{a^2}}\right]$; $a \ge b \ge c$		
Surface of Revolution	*For revolution about the x-axis:* $A = 2\pi \int_a^b f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ *For revolution about the y-axis:* $A = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$	*For revolution about the x-axis:* $A = 2\pi \int_\alpha^\beta r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ *For revolution about the y-axis:* $A = 2\pi \int_\alpha^\beta r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$	*Sphere:* $S = 4\pi r^2$	*For revolution about the x-axis:* $A_x = 2\pi \int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ *For revolution about the y-axis:* $A_y = 2\pi \int_a^b x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$	NA	NA

	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Volume	*Cube:* $V = s^3$ *Rectangular Prism:* $V = lwh$ *Cylinder:* $V = \pi r^2 h$ *Triangular Prism:* $V = Bh$ *Tetrahedron:* $V = \frac{1}{3} Bh$ *Pyramid:* $V = \frac{1}{3} Bh$ *Cone:* $V = \frac{1}{3} Bh = \frac{1}{3} \pi r^2 h$ *Sphere:* $V = \frac{4}{3}\pi r^3$ *Ellipsoid:* $V = \frac{4}{3}\pi abc$ $\int\int\int f(x,y,z)\, dx\, dy\, dz$	$\int\int\int f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta$	$\int\int\int f\begin{pmatrix}\rho\sin\varphi\cos\theta, \\ \rho\sin\varphi\ \sin\theta, \\ \rho\cos\varphi\end{pmatrix}$ … $\rho^2 \sin\varphi\ d\rho\, d\varphi\, d\theta$			*Ellipsoid:* $V = \frac{4}{3}\pi\sqrt{det(A^{-1})}$
Volume of Revolution	**Disk Method** $V = \int_a^b (area\ of\ circle)\ d(thickness)$ *Rotation about the x-axis:* $V = \int_a^b \pi\,[f(x)]^2 dx$ *Rotation about the y-axis:* $V = \int_c^d \pi x^2 dy$					
	Washer Method *Rotation about the x-axis:* $V = \int_a^b \pi\,\{[f(x)]^2 - [g(x)]^2\}\, dx$	$V = V_{Outer\ Disk} - V_{Inner\ Disk}$				
	Shell Method $V = \int_a^b (circumference)\ (hight)\ dx$ *Rotation about the y-axis:* $V = \int_a^b 2\pi x\, f(x)\, dx$ *Rotation about the x-axis:* $V = \int_c^d 2\pi y\, g(y)\, dy$					

	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Moments of Inertia	$I = \sum_{i=1}^{N} m_i\, r_i^2 = \int_0^a m\, r^2\, dr$	*NA*	*NA*		I $= \iiint_V \rho(\mathbf{r})\, d(\mathbf{r})^2\, dV(\mathbf{r})$	*(see Wikipedia)*
Center of Mass	$\boldsymbol{R} = \frac{1}{M}\sum_{i=1}^{N} m_i\, r_i$ *where* $M = \sum_{i=1}^{N} m_i$ *1D for Discrete:* $x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$	*2D for Discrete:* $M_y = \sum_{i=1}^{N} m_i\, x_i$ $M_x = \sum_{i=1}^{N} m_i\, y_i$ $\bar{x} = \frac{M_y}{M},\ \ \bar{y} = \frac{M_x}{M}$	*3D for Discrete:* $x_{cm} = \bar{x} = \frac{1}{M}\sum_{i=1}^{N} m_i\, x_i$ $y_{cm} = \bar{y} = \frac{1}{M}\sum_{i=1}^{N} m_i\, y_i$ $z_{cm} = \bar{z} = \frac{1}{M}\sum_{i=1}^{N} m_i\, z_i$	*3D for Continuous:* $\bar{x} = \frac{1}{M}\int_0^M x\, dm$ $\bar{y} = \frac{1}{M}\int_0^M y\, dm$ $\bar{z} = \frac{1}{M}\int_0^M z\, dm$ *where* $M = \int_0^M dm$ *and* $dm = \rho\, dz\, dy\, dx$	$\boldsymbol{R} = \frac{1}{M}\int \boldsymbol{r}\, dm$ $\boldsymbol{R} = \frac{1}{M}\iiint_V \rho(\boldsymbol{r})\, \boldsymbol{r}\, dV$ *Where* $\boldsymbol{r}$ *is distance from the axis of rotation, not origin.*	
Gradient	$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$	$\nabla f(\rho,\phi,z) = \frac{\partial f}{\partial \rho}\boldsymbol{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi + \frac{\partial f}{\partial z}\boldsymbol{e}_z$	$\nabla f(r,\theta,\phi) =$ $\frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta + \frac{1}{r\,\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi$		$(\nabla f(x)) \bullet \boldsymbol{v} = D_{\boldsymbol{v}} f(x)$ $\nabla f = \frac{\partial f_i}{\partial x_j}\boldsymbol{e}_i\boldsymbol{e}_j$ *where* $f = (f_1, f_2, f_3)$	
Line Integral	$\int_C f\, ds = \int_a^b f(\boldsymbol{r}(t))\lvert\boldsymbol{r}'(t)\rvert\, dt$	*NA*	*NA*		$\int_C \boldsymbol{F}(r) \bullet dr$ $= \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \bullet \boldsymbol{r}'(t)\, dt$	
Surface Integral	$\int_S f\, dS =$ $\iint_T f(\boldsymbol{x}(s,t)) \left\lvert \frac{\partial \boldsymbol{x}}{\partial s} \times \frac{\partial \boldsymbol{x}}{\partial t} \right\rvert ds\, dt$ *Where* $\boldsymbol{x}(s,t) =$ $(x(s,t), y(s,t), z(s,t))$ *and* $\left(\frac{\partial \boldsymbol{x}}{\partial s} \times \frac{\partial \boldsymbol{x}}{\partial t}\right) =$ $\left(\frac{\partial(y,z)}{\partial(s,t)}, \frac{\partial(z,x)}{\partial(s,t)}, \frac{\partial(x,y)}{\partial(s,t)}\right)$	*NA*	*NA*		$\int_S \boldsymbol{v} \bullet d\boldsymbol{S} =$ $\int_S (\boldsymbol{v} \bullet \boldsymbol{n})\, d\boldsymbol{S} =$ $\iint_T \boldsymbol{v}(\boldsymbol{x}(s,t)) \bullet \left(\frac{\partial \boldsymbol{x}}{\partial s} \times \frac{\partial \boldsymbol{x}}{\partial t}\right) ds\, dt$	