Harold's Business Calculus
Cheat Sheet
22 December 2022

Algebra Reference

Exponents		
Multiplication	$a^n a^m = a^{n+m}$	$\frac{a^n}{a^m} = a^{n-m} = \frac{1}{a^{m-n}}$
Power to a Power	$(a^n)^m = a^{nm}$	
Distributive	$(ab)^n = a^n b^n$	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
Zero Power	$a^0 = 1 \quad if\ a \neq 0$	
Power Sign Change	$a^{-n} = \frac{1}{a^n}$	$\frac{1}{a^{-n}} = a^n$
Negative Powers	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$	$a^{\frac{n}{m}} = \left(a^{\frac{1}{m}}\right)^n = (a^n)^{\frac{1}{m}}$

Radicals		
Convert to Power	$\sqrt[n]{a} = a^{\frac{1}{n}}$	$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
Root of a Root	$\sqrt[m]{\sqrt[n]{a}} = \sqrt[nm]{a}$	$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Logarithms		
Definition	Log ≡ Exponential	
	$log_b\, x = y \quad \equiv \quad x = b^y$	
Example	$log_5\, 125 = 3 \quad \equiv \quad 125 = 5^3$	
Common Log	$log\, x = log_{10}\, x$	*Base e assumed in pre-1955 math textbooks.* *Base 2 assumed in computer science textbooks.*
Natural Log	$ln\, x = log_e\, x$	*where* $e \approx 2.718281828\ldots$
Powers (x^2)	$log_b(x^r) = r\, log_b\, x$	$ln\, x^r = r\, ln\, x$
Multiplication (×)	$log_b(xy) = log_b\, x + log_b\, y$	$ln(xy) = ln\, x + ln\, y$
Division (÷)	$log_b\left(\frac{x}{y}\right) = log_b\, x - log_b\, y$	$ln\left(\frac{x}{y}\right) = ln\, x - ln\, y$
Zero (0) and One (1)	$log_b\, 1 = 0$	$log_b\, b = 1$
Inverse Functions	$log_b\, b^x = x$	$b^{log_b\, x} = x$
Change of Base	$log_b\, x = \frac{log_a\, x}{log_a\, b} = \frac{ln\, x}{ln\, b}$	**TI-84:** [MATH] + [A: logBASE( ] → log ( )

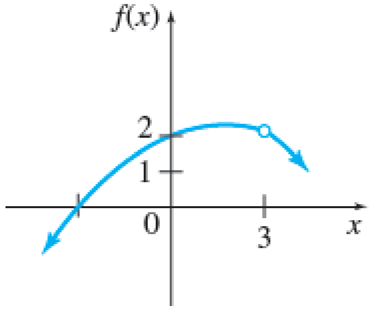
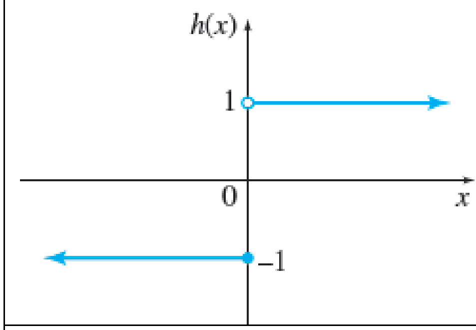
3.1 Limits

Property	(Map to Larson's 1-pager of common derivatives)
Definition of Limit	Let f be a function and let a and L be real numbers. If 1. as x takes values closer and closer (but not equal) to a on both sides of a, the corresponding values of $f(x)$ get closer and closer (and perhaps equal) to L; and 2. the value of $f(x)$ can be made as close to L as desired by taking values of x close enough to a; then L is the limit of $f(x)$ as x approaches a, written $$\lim_{x \to a} f(x) = L$$
Existence of Limits	The limit of f as x approaches a may not exist. 1. If $f(x)$ becomes infinitely large in magnitude (positive or negative) as x approaches the number a from either side, we write $$\lim_{x \to a} f(x) = \infty$$ or $$\lim_{x \to a} f(x) = -\infty$$ In either case the limit does not exist. 2. If $f(x)$ becomes infinitely large in magnitude (positive) as x approaches a from one side and infinitely large in magnitude (negative) as x approaches a from the other side, then $\lim_{x \to a} f(x)$ does not exist. 3. If $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = M$, and $L \neq M$, then $\lim_{x \to a} f(x)$ does not exist.
Limits at Infinity	$$\lim_{x \to \infty} \frac{1}{x^n} = 0$$ $$\lim_{x \to -\infty} \frac{1}{x^n} = 0$$
Finding Limits at Infinity	If $f(x) = \frac{p(x)}{q(x)}$, for polynomials $p(x)$ and $q(x), q(x) \neq 0$, $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ can be found as follows. 1. Divide $p(x)$ and $q(x)$ by the highest power of x in $q(x)$. 2. Use the rules for limits, including the rules for limits at infinity, $$\lim_{x \to \infty} \frac{1}{x^n} = 0$$ and $$\lim_{x \to -\infty} \frac{1}{x^n} = 0$$ to find the limit of the result from Step 1.

Rules for Limits

Rule	(Map to Larson's 1-pager of common derivatives)
Given	Let a, A, and B be real numbers, and let f and g be functions such that $$\lim_{x \to a} f(x) = A$$ and $$\lim_{x \to a} g(x) = B.$$
1. Constant (*c*)	If c is a constant, then $$\lim_{x \to a} c = c$$ and $$\lim_{x \to a}[c \cdot f(x)] = c \cdot \lim_{x \to a} f(x) = c \cdot A$$
2. Sum or Difference (+, −)	The limit of a sum or difference is the sum or difference of the limits. $$\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = A \pm B$$
3. Product (×)	The limit of products is the product of the limits. $$\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = A \cdot B$$
4. Quotient (÷)	The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero. $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{A}{B}$$ if $B \neq 0$.
5. Polynomial ($P(x)$)	If $p(x)$ is a polynomial, then $$\lim_{x \to a} p(x) = p(a)$$
6. Exponent (x^k)	For any real number k, $$\lim_{x \to a}[f(x)]^k = \left[\lim_{x \to a} f(x)\right]^k = A^k$$ provided that this limit exists.
7. Equivalent Functions (=)	$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$ If $f(x) = g(x)$ for all $x \neq a$.
8. Function Exponent	For any real number $b > 0$, $$\lim_{x \to a} b^{f(x)} = b^{\lim_{x \to a} f(x)} = b^A$$
9. Logorithm	For any real number b such that $0 < b < 1$ or $1 < b$, $$\lim_{x \to a}[log_b f(x)] = log_b \left[\lim_{x \to a} f(x)\right] = log_b A$$ if $A > 0$.

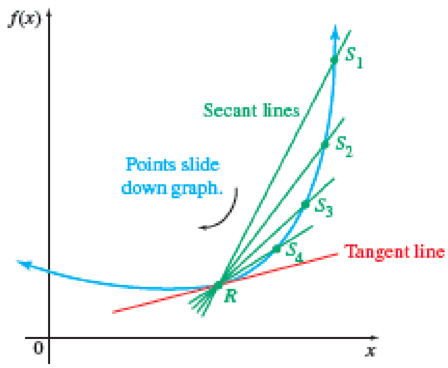
3.2 Continuity

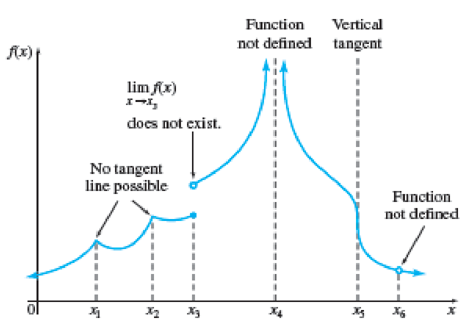
Term	Definition
Continuity at $x = c$   	A function f is **continuous** at $x = c$ if the following three condistions are satisfied: 1. $f(c)$ is defined, 2. $\lim_{x\to c} f(x)$ exists, and 3. $\lim_{x\to c} f(x) = f(c)$. If f is not continuous at c, it is **discontinuous** there.
Continuity on a Closed Interval	A function is **continuous on a closed interval** $[a, b]$ if 1. It is continuous on the open interval (a, b), 2. It is continuous from the right at $x = a$, and 3. It is continuous from the left at $x = b$.

3.3 Rates of Change

Term	Equation
Average Rate of Change	The **average rate of change** of $f(x)$ with respect to x for a function as x changes from a to b is $\dfrac{f(b) - f(a)}{b - a}$
Instantaneous Rate of Change	The **instantaneous rate of change** for a function f when $x = a$ is $\lim_{h\to 0} \dfrac{f(a + h) - f(a)}{h}$ or $\lim_{b\to a} \dfrac{f(b) - f(a)}{b - a}$ provided this limit exists.

3.4 Definition of the Derivative

<table>
<tr><th>Term</th><th colspan="3">Definition</th></tr>
<tr><td>Slope of the Tangent Line</td><td colspan="3">The tangent line of the graph of $y = f(x)$ at the point $(a, f(a))$ is the line through this point having slope
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
provided this limit exists.
If the limit does not exist, then there is no tangent at that point.</td></tr>
<tr><td>Derivative

</td><td colspan="3">The derivative of the function f at x is defined as
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
or
$$f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b - x}$$
provided this limit exists.

The derivative is the slope generating function at any point x.
It is usually set = 0 to find minimum (loss) and maximum (profit) values of $f(x)$.</td></tr>
<tr><td>Notations for the Derivative of
$y = f(x)$</td><td colspan="2">$\frac{dy}{dx} = y'$
$\frac{d}{dx}[f(x)] = f'(x)$</td><td>$f^{(n)}(x)$
$D_x[y]$</td></tr>
<tr><td rowspan="4">Equivalent Expressions for the Change in x</td><td>$x_2 - x_1$</td><td colspan="2">Useful for describing the equation of a line through two points.</td></tr>
<tr><td>$b - a$</td><td colspan="2">A way to write $x_2 - x_1$ without the subscripts.</td></tr>
<tr><td>Δx</td><td colspan="2">Useful for describing the change in x without referring to the individual points.</td></tr>
<tr><td>h</td><td colspan="2">A way to write Δx with just one symbol.</td></tr>
<tr><td>Equation of the Tangent Line</td><td colspan="3">The tangent line to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$ is given by the equation
$$y - f(x_1) = f'(x_1)(x - x_1),$$
provided $f'(x)$ exists.</td></tr>
<tr><td>Existence of the Derivative

</td><td colspan="3">The derivative exists when a function f satisfies all of the following conditions at a point.
1. f is continuous,
2. f is smooth, and
3. f does not have a vertical tangent line.

The derivative does not exist when any of the following conditions are true for a function at a point.
1. f is discontinuous,
2. f has a sharp corner, or
3. f has a vertical tangent line.</td></tr>
</table>

4. Derivative Formulas

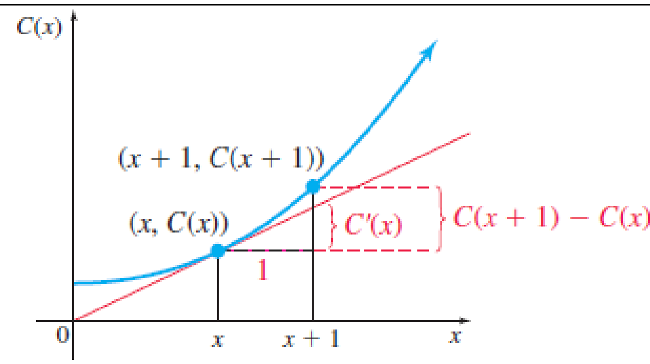
Rule	Formula
1. Chain Rule (🔗)	$\frac{d}{dx}[f \circ g(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$
2. Constant Rule (*c*)	$\frac{d}{dx}[c] = 0$ $(think\ \ \mathrm{c} = \mathrm{c}x^0 \rightarrow c0x^{-1} = 0\ \ after\ applying\ rule\ \#7)$
3. Constant Multiple Rule (*c*)	$\frac{d}{dx}[cf(x)] = cf'(x)$
4. Sum and Difference Rule (+, −)	$\frac{d}{dx}[f \pm g] = f' \pm g'$
5. Product Rule (×)	$\frac{d}{dx}[fg] = fg' + gf'$
6. Quotient Rule (÷)	$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$ $(same\ as\ \frac{f}{g} = \ fg^{-1}\ \ then\ apply\ rule\ \#4)$
7. Power Rule (x^n)	$\boldsymbol{\frac{d}{dx}[cx^n] = cnx^{n-1}}$
8. General Power Rule (x^n)	$\frac{d}{dx}[f^n] = nf^{n-1}\, f'$
9. Power Rule for $f(x) = x$	$\frac{d}{dx}[x] = 1$ $(think\ \ x = x^1 \rightarrow 1x^0 = 1\ \ after\ applying\ rule\ \#7)$
10. Natural Exponential Rule	$\frac{d}{dx}[e^x] = e^x$
11. General Natural Exponential Rule	$\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)} \cdot g'(x)$
12. Exponential Rule	$\frac{d}{dx}[a^x] = (ln\, a) \cdot a^x$
13. General Exponential Rule	$\frac{d}{dx}\left[a^{g(x)}\right] = (ln\, a) \cdot a^{g(x)} \cdot g'(x)$
14. Natural Logorithm Rule	$\frac{d}{dx}[ln\, x] = \frac{1}{x}$
15. General Natural Logorithm Rule	$\frac{d}{dx}[ln\, f(x)] = \frac{1}{f(x)} \cdot f'(x)$
16. Logorithm Rule	$\frac{d}{dx}[log_a\, x] = \frac{1}{(ln\, a)\, x}$
17. General Logorithm Rule	$\frac{d}{dx}[log_a\, f(x)] = \frac{1}{ln\, a} \cdot \frac{f'(x)}{f(x)}$

Equation of a Line

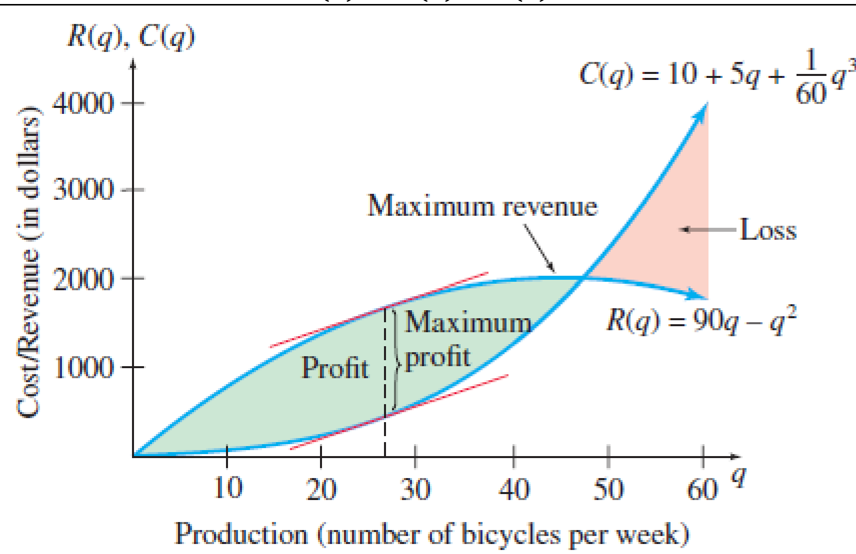
Form	Equation
Genreal Form	$ax + by + c = 0$
Slope-Intercept Form	$y = mx + b$
Point-Slope Form	$y - y_0 = m(x - x_0)$ $where\ m = f'(x_0)\ at\ point\ (x_0, y_0)$
Calculus Form	$y = f'(c)(x - c) + f(c)$
Slope	$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} \to \frac{dy}{dx} = f'(x)$

4.1 Derivative Applications

Application	Business Case
Average Cost	If the total cost to manufacture x items is given by $C(x)$, then the **average cost** per item is $\overline{C}(x) = C(x)/x$.
Marginal Average Cost	The **marginal average cost** is the derivative of the average cost function, $\overline{C}'(x)$.

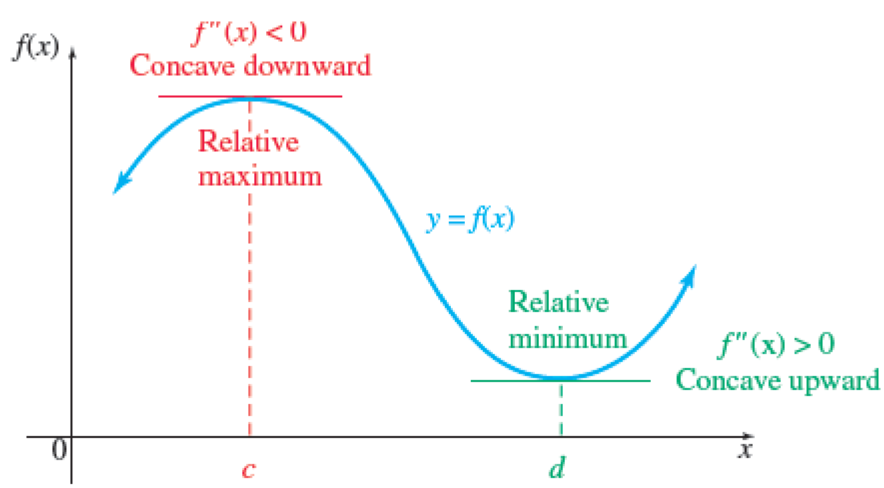


Profit	Profit equals total Revenue minus Cost or Expenses. $P(x) = R(x) - C(x)$



5. Graphing with Derivatives

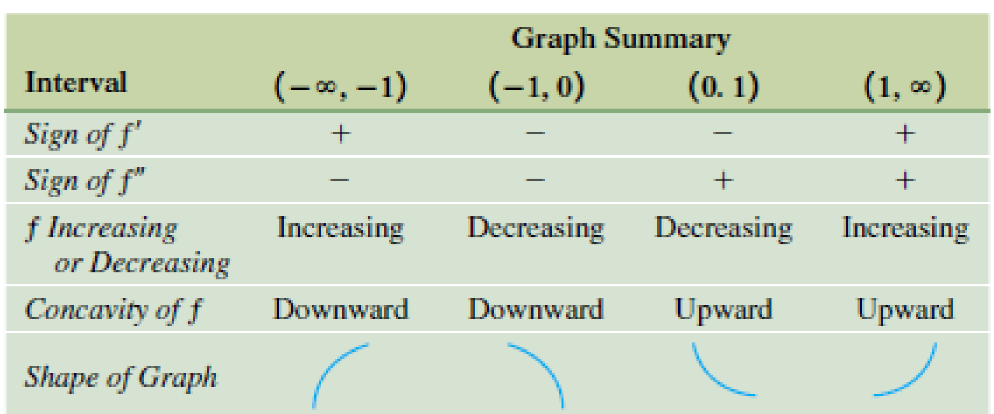
Term	Definition
Test for Increasing and Decreasing Functions	1. If $f'(x) > 0$, then f is increasing (slope up). ↗ 2. If $f'(x) < 0$, then f is decreasing (slope down). ↘ 3. If $f'(x) = 0$, then f is constant (zero slope). →
Critical Numbers	The **critical numbers** for a function f are those numbers c in the domain of f for which $f'(c) = 0$ or $f'(c)$ does not exist.
Critical Point	A **critical point** is a point whose x-coordinate is the critical number c and whose y-coordinate is $f(c)$.
First Derivative (Slope Formula)	$f'(x) = 0$ *finds critical points (min. and max.).* *Don't forget to check the boundaries:* $f(a)$ *and* $f(b)$.
First Derivative Test	1. If $f'(x)$ changes from – to + at *c*, then f has a *relative minimum* at $(c, f(c))$. 2. If $f'(x)$ changes from + to - at *c*, then f has a *relative maximum* at $(c, f(c))$. 3. If $f'(x)$, is + *c* + or – *c* –, then $f(c)$ is neither.
Test for Concavity	1. If $f''(x) > 0$ for all x, then the graph is concave up. ∪ 2. If $f''(x) < 0$for all x, then the graph is concave down. ∩
Second Deriviative Test Let *f '(c)*=0, and *f "(x)* exists, then	1. If $f''(x) > 0$, then f has a relative minimum at $(c, f(c))$. 2. If $f''(x) < 0$, then f has a relative maximum at $(c, f(c))$. 3. If $f''(x) = 0$, then the test fails (See 1^{st} derivative test). If $f''(x) \to +$, then cup up ∪ (min.) If $f''(x) \to -$, then cup down ∩ (max.)
Points of Inflection Change in concavity	If $(c, f(c))$ is a point of inflection of $f(x)$, then either 1. $f''(c) = 0$ or 2. $f''(x)$ does not exist at *x* = *c*.



5.4 Curve Sketching

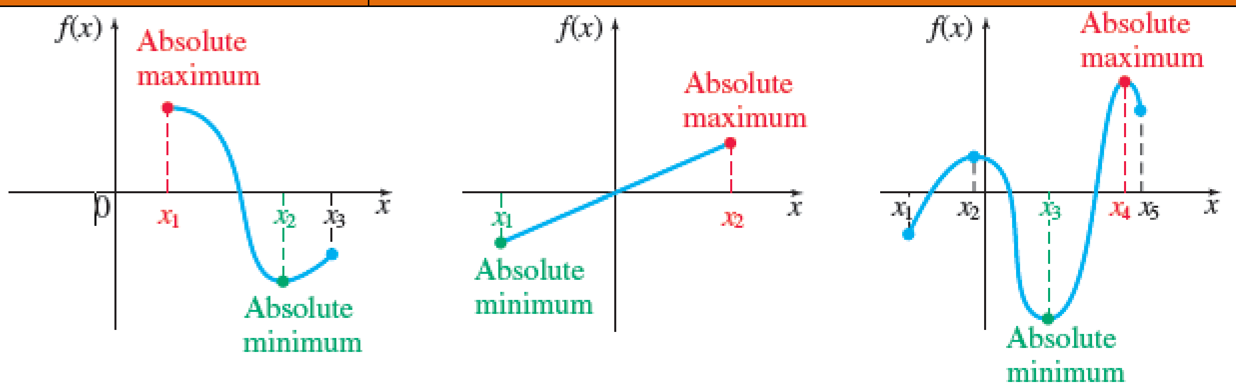
Step	Description
To sketch the graph of a function f:	
1.	Consider the domain of the function, and note any restrictions. (That is, avoid dividing by 0, taking a square root, or any even root, of a negative number, or taking the logarithm of 0 or a negative number.)
2.	Find the y-intercept (if it exists) by substituting $x = 0$ into $f(x)$. Find any x-intercepts by solving $f(x) = 0$ if this is not too difficult.
3.	(a) If f is a rational function, find any vertical asymptotes (VA) by investigating where the denominator is 0, and find any horizontal asymptotes (HA) by finding the limits as $x \to \infty$ and $x \to -\infty$. (b) If f is an exponential function, find any horizontal asymptotes (HA); If f is a logarithmic function, find any vertical asymptotes (VA).
4.	Investigate symmetry. If $f(-x) = f(x)$, the function is **even**, so the graph is symmetric about the y-axis. If $f(-x) = -f(x)$,, the function is **odd**, so the graph is symmetric about the origin.
5.	Find $f'(x)$. Locate any critical points by solving the equation $f'(x) = 0$ and determining where $f'(x)$ does not exist, but $f(x)$ does. Find any relative extrema and determine where f is increasing or decreasing.
6.	Find $f''(x)$. Locate potential inflection points by solving the equation $f''(x) = 0$ and determining where $f''(x)$ does not exist. Determine where f is concave upward or concave downward.
7.	Plot the x and y intercepts, the critical points, the inflection points, the asymptotes, and other points as needed. Take advantage of any symmetry found in Step 4.
8.	Connect the points with a smooth curve using the correct concavity, being careful not to connect points where the function is not defined.
9.	Check your graph using a graphing calculator or desmos. If the picture looks very different from what you've drawn, see in what ways the picture differs and use that information to help find your mistakes.

Example Chart



Graph Summary				
Interval	$(-\infty, -1)$	$(-1, 0)$	$(0. 1)$	$(1, \infty)$
Sign of f'	$+$	$-$	$-$	$+$
Sign of f''	$-$	$-$	$+$	$+$
f Increasing or Decreasing	Increasing	Decreasing	Decreasing	Increasing
Concavity of f	Downward	Downward	Upward	Upward
Shape of Graph				

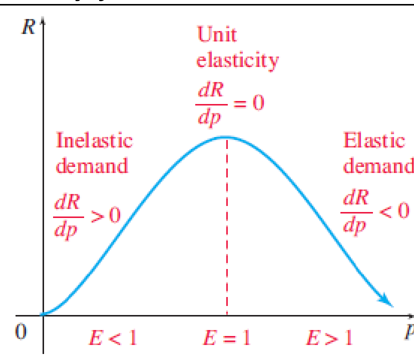
6.1 Absolute Extrema

Term	Definition
	
Absolute Maximum	Let *f* be a function defined on some interval. Let c be a number in the interval. Then $f(c)$ is the **absolute maximum** of f on the interval if $f(x) \leq f(c)$ for every x in the interval.
Absolute Minimum	Let *f* be a function defined on some interval. Let c be a number in the interval. Then $f(c)$ is the **absolute minimum** of f on the interval if $f(x) \geq f(c)$ for every x in the interval.
Absolute Extremum (Extrima)	A function f has an **absolute extremum** (plural: **extrema**) at c if it has either an absolute maximum or an absolute minimum there.
Extreme Value Theorem	A function f that is continuous on a closed interval [*a*, *b*] will have both an absolute maximum and an absolute minimum on the interval.
Finding Absolute Extrema	To find absolute extrema for a function f continuous on a closed interval [*a*, *b*]: 1. Find all critical numbers for f in (*a*, *b*). 2. Evaluate f for all critical numbers in (*a*, *b*). 3. Evaluate f for the *endpoints a* and *b* of the interval [*a*, *b*]. 4. The largest value found in Step 2 or 3 is the absolute maximum for f on [*a*, *b*], and the smallest value found is the absolute minimum for f on [*a*, *b*].
Critical Point Theorem	Suppose a function f is continuous on an interval *I* and that f has exactly one critical number in the interval *I*, located at $x = c$. If f has a relative maximum at $x = c$, then this relative maximum is the absolute maximum of f on the interval *I*. If f has a relative minimum at $x = c$, then this relative minimum is the absolute minimum of f on the interval I.

6.2 Applications of Extrema

Step	Description
Solving an Applied Extrema Problem	
1.	Read the problem carefully. Make sure you understand what is given and what is unknown.
2.	If possible, sketch a diagram. Label the various parts.
3.	Decide which variable must be maximized or minimized. Express that variable as a function of one other variable.
4.	Find the domain of the function.
5.	Find the critical points for the function from Step 3.
6.	If the domain is a closed interval, evaluate the function at the endpoints and at each critical number to see which yields the absolute maximum or minimum. If the domain is an open interval, apply the critical point theorem when there is only one critical number. If there is more than one critical number, evaluate the function at the critical numbers and find the limit as the endpoints of the interval are approached to determine if an absolute maximum or minimum exists at one of the critical points.

6.3 Further Business Application

Economic Lot Size, Economic Order Quantity, Elasticity of Demand	
Elasticity of Demand	Let $q = f(p)$, where q is demand at a price p. The **elasticity of demand** is $$E = -\frac{p}{q} \cdot \frac{dq}{dp}.$$ *Demand is* ***inelastic*** *if* $E < 1$. *Demand is* ***elastic*** *if* $E > 1$. *Demand has* ***unit elasticity*** *if* $E = 1$.
	
Revenue and Elasticity	1. If the demand is inelastic, total revenue increases as price increases. 2. If the demand is elastic, total revenue decreases as price increases. 3. Total revenue is maximized at the price where demand has unit elasticity.

6.4 Implicit Differentiation

<table>
<tr><th>Term</th><th colspan="2">Definition</th></tr>
<tr><td>Implicit Differentiation</td><td colspan="2">To find dy/dx for an equation containing x and y:

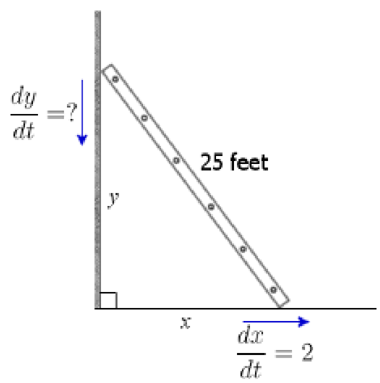
1. Differentiate on both sides of the equation with respect to x, keeping in mind that y is assumed to be a function of x.

2. Using algebra, place all terms <u>with</u> dy/dx on one side of the equals sign and all terms <u>without</u> dy/dx on the other side.

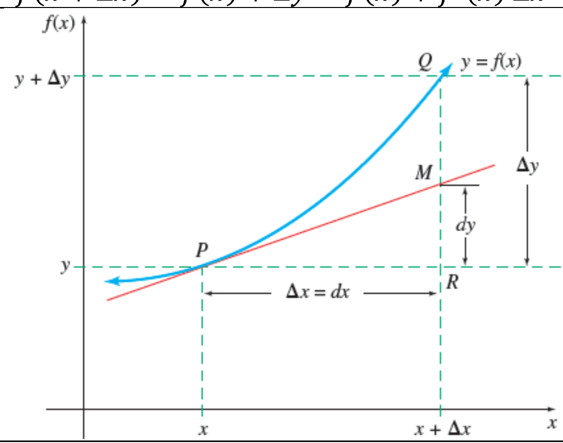
3. Factor out dy/dx, and then divide to solve for dy/dx.</td></tr>
<tr><td rowspan="9">Example

$\frac{d}{dx}[cx^n y^m = k]$</td><td>$c[(x^n)(y^m)' + (y^m)(x^n)'] = 0$</td><td>Product Rule</td></tr>
<tr><td>$c[(x^n)(my^{m-1}y') + (y^m)(nx^{n-1}x')] = 0$</td><td>Chain Rule</td></tr>
<tr><td>$cmx^n y^{m-1}y' + cny^m x^{n-1} = 0$</td><td>Distribute. $x' = 1$</td></tr>
<tr><td>$cmx^n y^{m-1}y' = -cnx^{n-1}y^m$</td><td>Subtract</td></tr>
<tr><td>$y' = -\frac{cnx^{n-1}y^m}{cmx^n y^{m-1}}$</td><td>Divide</td></tr>
<tr><td>$y' = -\frac{nx^{n-1}y^m}{mx^n y^{m-1}}$</td><td>Cancel common term (c)</td></tr>
<tr><td>$y' = -\frac{n}{m}x^{n-1-n}y^{m-(m-1)}$</td><td>Bring powers to numerator</td></tr>
<tr><td>$y' = -\frac{n}{m}x^{-1}y^1$</td><td>Simplify</td></tr>
<tr><td>$y' = -\frac{ny}{mx}$</td><td>Cleanup negative and unit exponents</td></tr>
</table>

6.5 Related Rates

Term	Definition
Solving a Related Rates Problem	1. Identify all given quantities, as well as the quantities to be found. Draw a sketch when possible. 2. Write an equation relating the variables of the problem. 3. Use implicit differentiation to find the derivative of both sides of the equation in Step 2 with respect to time (t). 4. Solve for the derivative giving the unknown rate of change and substitute the given values.
Example 	Steps to solve: 1. Identify the known variables and rates of change. $x = 15\,m$ $y = 20\,m$ $x' = 2\frac{m}{s}$ $y' = ?\,\frac{m}{s}$ 2. Construct an equation relating these quantities. $x^2 + y^2 = r^2$ 3. Differentiate both sides of the equation. $2xx' + 2yy' = 0$ 4. Solve for the desired rate of change. $y' = -\frac{x}{y}x'$ 5. Substitute the known rates of change and quantities into the equation. $y' = -\frac{15}{20} \bullet 2 = \frac{3}{2}\frac{m}{s}$

6.6 Differentials: Linear Approximation

Differentials	Formula
Differentials	For a function $y = f(x)$ whose derivative exists, the differential of x, written dx, is an arbitrary real number (usually small compared with x); the differential of y, written dy, is the product of $f'(x)$ and dx, or $$dy = f'(x)\,dx$$ or $$\Delta y = f'(x)\,\Delta x$$
Relative Error	$\text{Relative Error} = \frac{\Delta f}{f}$ in %
Linear Approximation	Let f be a function whose derivative exists. For small nonzero values of Δx, $$dy \approx \Delta y$$ and $$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)\,dx$$ or $$f(x + \Delta x) \approx f(x) + \Delta y \approx f(x) + f'(x)\,\Delta x$$
	
Example	Solve for $\sqrt[4]{82}$ Rewrite as $f(x) = \sqrt[4]{x}$ $f(x + \Delta x) = f(81 + 1)$ $\approx f(x) + f'(x)\,\Delta x$ $$f'(x) = \frac{1}{4}\left(x^{-\frac{3}{4}}\right) = \left(\frac{1}{4(x)^{\frac{3}{4}}}\right)$$ $$f'(x) = \sqrt[4]{81} + \left(\frac{1}{4(81)^{\frac{3}{4}}}\right)$$ $$= 3 + \left(\frac{1}{4(3)^3}\right)$$ $$= 3 + \frac{1}{108} = \frac{325}{108} \approx 3.009259$$ $Estimate = 3.009259$ $Actual = 3.009217$

7.1 Antiderivative / Integration

Rule	Formulas
Antiderivative	If $F'(x) = f(x)$, then $F(x)$ is an antiderivative of $f(x)$.
Notation	$$\int f(x)\,dx = F(x) + C$$
Concept	If $F(x)$ and $G(x)$ are both antiderivatives of a function $f(x)$ on an interval, then there is a constant *C* such that $$F(x) - G(x) = C$$ (Two antiderivatives of a function can differ only by a constant.) The arbitrary real number *C* is called an integration constant.
Indefinite Integral	If $F'(x) \;= f(x)$, then $$\int f(x)\,dx = F(x) + C$$ for any real number *C*.
Power Rule (x^n)	For any real number $n \neq -1$, $$\int cx^n\,dx = c\frac{x^{n+1}}{n+1} + C$$
Constant Multiple Rule (c)	$$\int c \cdot f(x)\,dx = c\int f(x)\,dx$$
Sum or Difference Rule (+, $-$)	$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$
Indefinite Integrals of Exponential Functions (e^x)	$$\int e^x\,dx = e^x + C$$ $$\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$$ For $a > 0, a \neq 1$: $$\int a^x\,dx = \frac{a^x}{\ln a} + C$$ $$\int a^{kx}\,dx = \frac{a^{kx}}{k(\ln a)} + C\,, k \neq 0$$
Natural Exponential Rule (e^f)	$$\int ke^f\,dx = k\frac{e^f}{f'} + C$$
Indefinite Integral of $f(x) = x^{-1}$	$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$$ $$\int \frac{dCabin}{Cabin} = Log\ Cabin\ by\ the\ sea$$

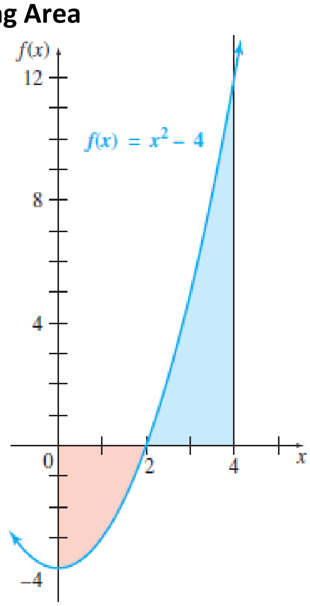
7.2 Integral Substitution

Method	Formula		
Substitution	Each of the following forms can be integrated using the substitution $u = f(x)$.		
		Form of the Integral	**Result**
	1.	$\int [f(x)]^n f'(x)\, dx, n \neq -1$	$\int u^n\, dx = \frac{u^{n+1}}{n+1} + C = \frac{[f(x)]^{n+1}}{n+1} + C$
	2.	$\int \frac{f'(x)}{f(x)} dx$	$\int \frac{1}{u}\, dx = ln\lvert u\rvert + C = ln\lvert f(x)\rvert + C$
	3.	$\int e^{f(x)} f'(x)\, dx$	$\int e^u\, dx = e^u + C = e^{f(x)} + C$
Substitution Method	In general there are three cases. We **choose u** to be one of the following: 1. the quantity under a root or raised to a power; 2. the quantity in the denominator; 3. the exponent on e. Always capture the constant in u, such as $u = g(x) \pm c$. Remember that some integrands may need to be rearranged to fit one of these cases.		
	$u = \underline{\ part\ of\ f(x)\ (see\ above)\ }$ $du = \underline{\ remaining\ part\ of\ f(x)\ }\ dx$		

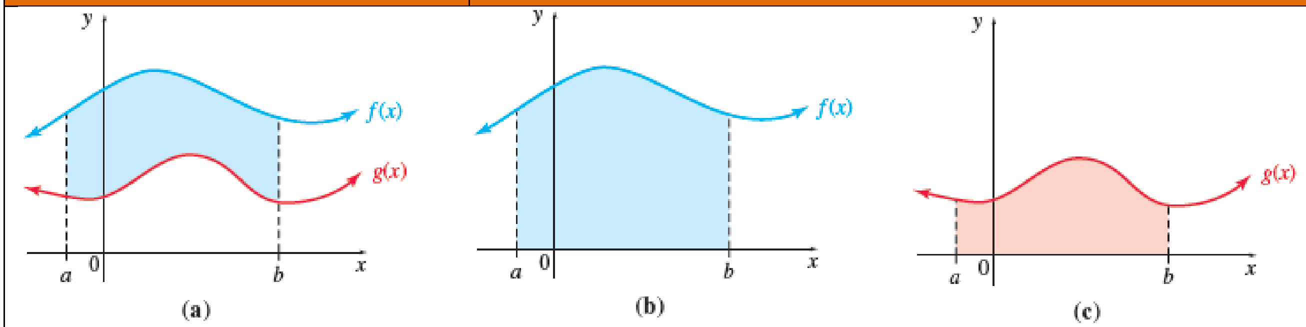
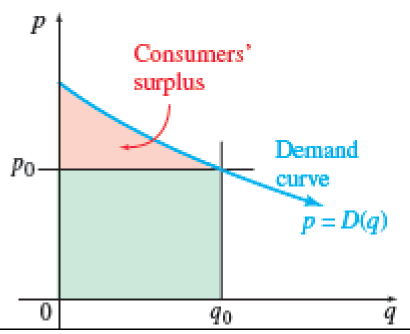
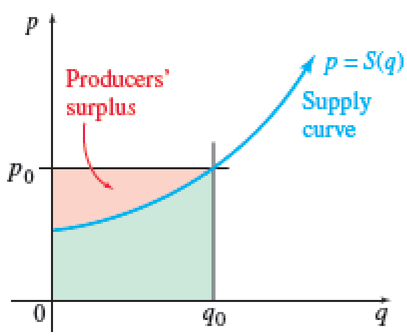
7.3 Area and the Definite Integral

Definition	Formula
The Definite Integral	If f is defined on the interval $[a, b]$, the definite integral of f from a to b is given by $$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\, \Delta x,$$ provided the limit exists, where $$\Delta x = \frac{(b-a)}{n}$$ and x_i is any value of x in the ith interval. aka Riemann sum.
Total Change in $F(x)$	If $f(x)$ gives the rate of change of $F(x)$ for x in $[a, b]$, then the total change in $F(x)$ as x goes from a to b is given by $$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\, dx.$$

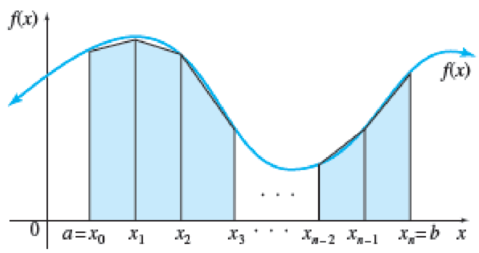
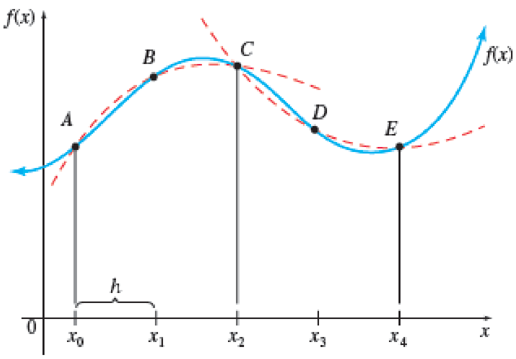
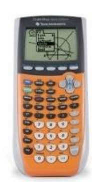
7.4 The Fundamental Theorem of Calculus

Definition	Formula
Fundamental Theorem of Calculus	Let f be continuous on the interval [*a, b*], and let F be any antiderivative of f. Then $$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big\|_a^b.$$
1. Constant Multiple of a Function (c)	$$\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$$
2. Sum or Difference of Functions (+, −)	$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$
3. Same Bounds	$$\int_a^a f(x) = 0$$
4. Split Bounds	$$\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)$$
5. Swap Bounds	$$\int_a^b f(x) = -\int_b^a f(x)$$
Finding Area 	In summary, to find the area bounded by $f(x), x = a, x = b$, and the x-axis, use the following steps. 1. Sketch a graph. 2. Find any x-intercepts of $f(x)$ in [*a, b*]. These divide the total region into subregions. 3. The definite integral will be positive for subregions above the x-axis and negative for subregions below the x-axis. Use separate integrals to find the (positive) areas of the subregions. 4. The total area is the sum of the areas of all of the subregions.

7.5 The Area Between Two Curves

Definition	Formula
	
Area Between Two Curves	If f and g are continuous functions and $f(x) \geq g(x)$ on [*a, b*], then the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is given by $$\int_a^b [f(x) - g(x)]\, dx.$$
Consumers' Surplus 	If $D(q)$ is a demand function with equilibrium price p_0 and equilibrium demand q_0, then **Customers' Surplus** is given by $$\int_0^{q_0} [D(q) - p_0]\ dq.$$
Producers' Surplus 	If $S(q)$ is a supply function with equilibrium price p_0 and equilibrium supply q_0, then **Producer's Surplus** is given by $$\int_0^{q_0} S(q)\ dq.$$

7.6 Numerical Integration

Rule	Formula
Trapezoidal Rule  	Let f be a continuous function on [*a, b*] and let [*a, b*] be divided into *n* equal subintervals by the points $a = x_0, x_1, x_2, \ldots, x_n = b$. Then, by the **trapezoidal rule**, $$\int_a^b f(x)\,dx \approx$$ $$\left(\frac{b-a}{n}\right)\left[\frac{1}{2}f(x_0) + f(x_1) + f(x_3) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]$$ and $$x_i = a + i\left(\frac{b-a}{n}\right)$$
Simpson's Rule  	Let f be a continuous function on [*a, b*] and let [*a, b*] be divided into *n* equal subintervals by the points $a = x_0, x_1, x_2, \ldots, x_n = b$. Then by **Simpson's rule**, $$\int_a^b f(x)\,dx \approx$$ $$\left(\frac{b-a}{3n}\right)[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots$$ $$+ 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$ Where n is even and $$x_i = a + i\left(\frac{b-a}{n}\right)$$
TI-84 	[MATH] fnInt(f(x),x,a,b), [MATH] [1] [ENTER] Example: [MATH] fnInt(x^2,x,0,1) $$\int_0^1 x^2\,dx = \frac{1}{3}$$
TI-Nspire CAS 	[MENU] [4] Calculus [3] Integral [TAB] [TAB] [X] [^] [2] [TAB] [TAB] [X] [ENTER] Shortcut: [ALPHA] [WINDOWS] [4]

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- All highlighted formulas copied from chapters 3–7 of "Calculus with Applications", 11th Edition (Global Edition), by Margaret L. Lial, Raymond N. Greenwell, and Nathan P. Ritchey, Pearson, 2017.
- Used in MATH 1325 Calculus for Business, Collin College, McKinney, Texas.

