

**AP<sup>®</sup> CALCULUS BC  
2006 SCORING GUIDELINES**

**Question 6**

The function  $f$  is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n nx^n}{n+1} + \cdots$$

for all real numbers  $x$  for which the series converges. The function  $g$  is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots$$

for all real numbers  $x$  for which the series converges.

- (a) Find the interval of convergence of the power series for  $f$ . Justify your answer.  
 (b) The graph of  $y = f(x) - g(x)$  passes through the point  $(0, -1)$ . Find  $y'(0)$  and  $y''(0)$ . Determine whether  $y$  has a relative minimum, a relative maximum, or neither at  $x = 0$ . Give a reason for your answer.

(a) 
$$\left| \frac{(-1)^{n+1} (n+1)x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n nx^n} \right| = \frac{(n+1)^2}{(n+2)(n)} \cdot |x|$$

$$\lim_{n \rightarrow \infty} \frac{(n+1)^2}{(n+2)(n)} \cdot |x| = |x|$$

The series converges when  $-1 < x < 1$ .

When  $x = 1$ , the series is  $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

When  $x = -1$ , the series is  $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

Thus, the interval of convergence is  $-1 < x < 1$ .

(b)  $f'(x) = -\frac{1}{2} + \frac{4}{3}x - \frac{9}{4}x^2 + \cdots$  and  $f'(0) = -\frac{1}{2}$ .

$$g'(x) = -\frac{1}{2!} + \frac{2}{4!}x - \frac{3}{6!}x^2 + \cdots$$
 and  $g'(0) = -\frac{1}{2}$ .

$$y'(0) = f'(0) - g'(0) = 0$$

$$f''(0) = \frac{4}{3} \text{ and } g''(0) = \frac{2}{4!} = \frac{1}{12}$$

$$\text{Thus, } y''(0) = \frac{4}{3} - \frac{1}{12} > 0.$$

Since  $y'(0) = 0$  and  $y''(0) > 0$ ,  $y$  has a relative minimum at  $x = 0$ .

5 : {  
 1 : sets up ratio  
 1 : computes limit of ratio  
 1 : identifies radius of convergence  
 1 : considers both endpoints  
 1 : analysis/conclusion for both endpoints

4 : {  
 1 :  $y'(0)$   
 1 :  $y''(0)$   
 1 : conclusion  
 1 : reasoning

NO CALCULATOR ALLOWED

Work for problem 6(a)

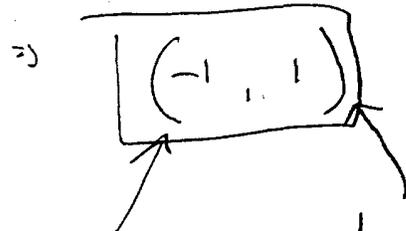
$$\lim_{n \rightarrow \infty} \frac{(-1)^{n+1} (n+1) x^{n+1}}{n+2}$$


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$$\lim_{n \rightarrow \infty} \frac{(-1)^{n+1} \cdot (n+1) \cdot x^{n+1}}{(-1)^n \cdot n \cdot x^n \cdot (n+2)}$$

$$= \lim_{n \rightarrow \infty} (x)$$

$\rightarrow R = 1$



$(-1)^{2n} \frac{n}{n+1}$  diverges due to the  $n^{\text{th}}$  term test

diverges due to the  $n^{\text{th}}$  term test

$$\lim_{n \rightarrow \infty} \frac{(-1)^n \cdot n \cdot 1^n}{n+1} \neq 0$$

$$\lim_{n \rightarrow \infty} \frac{(-1)^n \cdot n \cdot (1)^n}{n+1} \neq 0$$

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Continue problem 6 on page 15.

Work for problem 6(b)

$g(x) = \cos x$

$$y' = \frac{(-1)^n \cdot n \cdot x^{n-1}}{(n+0)(n-1)} - \frac{(-1)^n \cdot x^{n-1}}{(2n)!(n+1)} \Rightarrow y'(0) = 0$$

$$f'(x) = -\frac{1}{2} + \frac{4}{3}x \dots \Rightarrow f'(0) = -\frac{1}{2} \Rightarrow y'(0) = 0$$

$$g'(x) = -\frac{1}{2} + \frac{2}{4}x \dots \Rightarrow g'(0) = -\frac{1}{2}$$

$$f''(x) = \frac{4}{3} \Rightarrow y''(0) = \frac{4}{3} - \frac{1}{2} = \frac{15}{12} = \frac{5}{4}$$

$$g''(x) = \frac{1}{2}$$

has a relative min at  $x=0$  because the derivative shows there is a critical point at  $x=0$  and the 2nd derivative shows a positive concavity meaning the function values are decreasing up till  $x=0$  and increasing after therefore **STOP** showing a relative maximum

END OF EXAM

THE FOLLOWING INSTRUCTIONS APPLY TO THE COVERS OF THE SECTION II BOOKLET.

- MAKE SURE YOU HAVE COMPLETED THE IDENTIFICATION INFORMATION AS REQUESTED ON THE FRONT AND BACK COVERS OF THE SECTION II BOOKLET.
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NO CALCULATOR ALLOWED

Work for problem 6(a)

$$\frac{(n+1)x^{(n+1)}}{(n+1)+1} \cdot \frac{n+1}{n+1} = \frac{(n^2+2n+1)x}{(n^2+n)} = 3x$$

$$-1 < 3x < 1$$

$$\boxed{-\frac{1}{3} < x < \frac{1}{3}}$$

\* when  $-\frac{1}{3} < x < \frac{1}{3}$ , the power series for  $f$  converges.

$$\frac{(-1)^n n \left(-\frac{1}{3}\right)^n}{n+1} \rightarrow \text{diverges}$$

$$\frac{(-1)^n n \left(\frac{1}{3}\right)^n}{n+1} \rightarrow \text{converges}$$

↕  
alternating

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Work for problem 6(b)

$$y' = f'(x) - g'(x)$$

$$y'(0) = -\frac{1}{2} + \frac{1}{2} = \boxed{0}$$

$$y'' = f''(x) - g''(x)$$

$$y''(0) = \frac{4}{3} - \frac{1}{12} =$$

$$\frac{16}{12} - \frac{1}{12} = \boxed{\frac{15}{12}}$$

$$f'(x) = -\frac{1}{2} + \frac{4x}{3} \dots$$

$$f''(x) = \frac{4}{3}$$

$$g'(x) = -\frac{1}{2} + \frac{1}{12}x$$

$$g''(x) = \frac{1}{12}$$

$y$  has a relative minimum at  $x=0$  because the derivative of  $y$  at  $x=0$  is 0 and the second derivative of  $y$  at  $x=0$  is  $>0$ , meaning the graph of  $y$  is concave up at this point.

STOP

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Work for problem 6(a)

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n x^n}{n+1}$$

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1) x^{n+1}}{n+1} \cdot \frac{n+1}{(-1)^n n x^n} \right|$$

$$L = \lim_{n \rightarrow \infty} \left| (-1) x \cdot \frac{(n+1)^2}{(n+1)(n)} \right|$$

$$L = x$$

$$L < 1$$

$$x < 1$$

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NO CALCULATOR ALLOWED

Work for problem 6(b)

$$y = f(x) - g(x)$$

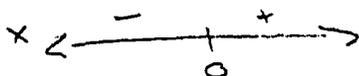
$$y'(x) = \left[ -\frac{1}{2} + \frac{4x}{3} + \dots \right] - \left[ -\frac{1}{2} + \frac{2x}{4!} + \dots \right]$$

$$y'(0) = -\frac{1}{2} - \left(-\frac{1}{2}\right)$$

$$y'(0) = 0$$

$$y''(x) = \left[ \frac{4}{3} - \frac{18}{4}x + \dots \right] - \left[ \frac{2}{24} - \frac{6x}{6!} + \dots \right]$$

$$y''(0) = \frac{4}{3} - \frac{1}{12} = \frac{15}{12}$$



relative minimum

STOP

END OF EXAM

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**AP<sup>®</sup> CALCULUS BC**  
**2006 SCORING COMMENTARY**

**Question 6**

**Overview**

This problem dealt with power series. Students were given the power series expansions of two functions,  $f$  and  $g$ . In part (a) they were asked to find the interval of convergence of the power series for  $f$ . Part (b) dealt with the graph of  $y = f(x) - g(x)$ . Students had to know how to read or compute the values of the first and second derivatives of  $y$  at  $x = 0$  from the series for  $f$  and  $g$ . They then needed to use this information to describe the nature of the critical point of  $y$  at  $x = 0$ .

**Sample: 6A**

**Score: 9**

The student earned all 9 points. In part (b) the student restarts the problem on the third line and earned all points.

**Sample: 6B**

**Score: 6**

The student earned 6 points: 2 points in part (a) and 4 points in part (b). In part (a) the student sets up an incorrect Ratio Test and does not compute a limit. The student earned the 2 points by stating the correct interval of convergence for the ratio and testing the endpoints. In part (b) the student calculates  $y'(0)$  and  $y''(0)$  and arrives at the correct conclusion with good reasoning.

**Sample: 6C**

**Score: 4**

The student earned 4 points: 1 point in part (a) and 3 points in part (b). In part (a) the student sets up the ratio correctly but does not identify the correct interval of convergence. In part (b) the student correctly calculates  $y'(0)$  and  $y''(0)$  and arrives at a correct conclusion. However, the student does not include a reason for the answer.