



AP Calculus BC 1999 Sample Student Responses

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4. The function f has derivatives of all orders for all real numbers x . Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

$$T_3(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 + -\frac{4}{3}(x-2)^3$$

$$\begin{aligned} f(1.5) &\approx -3 + 5(1.5-2) + \frac{3}{2}(1.5-2)^2 - \frac{4}{3}(1.5-2)^3 \\ &= -4.958 \end{aligned}$$

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- (b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \leq 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.

$$R_3(1.5) = \frac{f^{(4)}(z)}{4!} (1.5-2)^4 \quad \text{for some } z, 1.5 \leq z \leq 2.$$

$$\text{Thus, } R_3(1.5) \leq \frac{3}{4!} (1.5-2)^4 = .0078125$$

$$\begin{aligned} \text{Thus, } -4.958 - .0078125 &\leq f(1.5) \leq -4.958 + .0078125 \\ -4.966 &\leq f(1.5) \leq -4.950 \end{aligned}$$

$$\text{Thus, } f(1.5) \neq -5$$

- (c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

$$P(x) = -3 + 5x^2 + \frac{3}{2}x^4$$

Since the coefficient of x is 0, $\frac{g'(0)}{1!} = 0$, so $g'(0) = 0$.
Since the coefficient of x^2 is 5, $\frac{g''(0)}{2!} = 5$, so $g''(0) = 10$.

Thus, since $g''(0)$ is positive and $g'(0) = 0$, $P(x)$ must have a relative minimum at $x = 0$ by the second derivative test.

4. The function f has derivatives of all orders for all real numbers x . Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

$$\begin{aligned} f(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 \\ &= -3 + 5(x-2) + \frac{3}{2!}(x-2)^2 - \frac{8}{3!}(x-2)^3 \\ f(1.5) &= -4.958 \end{aligned}$$

- (b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \leq 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.

$$|\text{error}| \leq a_{n+1}$$

$$|\text{error}| \leq a_4$$

$$|\text{error}| \leq \frac{3}{4!}(x-2)^4$$

$$|\text{error}| \leq \frac{3}{24}(1.5-2)^4$$

$$|\text{error}| \leq .0078$$

The truncation error is no greater than .0078

$$\therefore -4.958 - .0078 < f(1.5) \leq -4.958 + .0078$$

$$-4.9658 \leq f(1.5) \leq -4.9502$$

$$\therefore f(1.5) \neq -5$$

- (c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

$$P(x) = -3 + 5(x^2 + 2 - 2) + \frac{3}{2!}(x^2 + 2 - 2)^2$$

$$= -3 + 5x^2 + \frac{3}{2!}x^4$$

$$P'(x) = 10x + 6x^3$$

$$0 = 2x(5 + 3x^2)$$

$$x = 0$$

$$\begin{array}{c} - \quad | \quad + \\ \hline 0 \end{array} \quad P'(x) = g'(x)$$

$\therefore g(x)$ has a relative minimum at $x = 0$

4. The function f has derivatives of all orders for all real numbers x . Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

$$f(x) = -3 + 5(x-2) + \frac{3(x-2)^2}{2!} - \frac{8(x-2)^3}{3!}$$

$$f(1.5) = -3 + 5(1.5-2) + \frac{3(1.5-2)^2}{2!} - \frac{8(1.5-2)^3}{3!}$$

$$f(1.5) = -4.958$$

- (b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \leq 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.

$$f^{(3)}(x) > f^{(4)}(x)$$

$$\frac{-8(-.5)^3}{3!} > \frac{f^{(4)}(-.5)^4}{4!}$$

$$1.667 > .002604 (f^{(4)})$$

since there is an error of at least .002604, $f(1.5) \neq -5$.

- (c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

$$g(x) = f(x^2 + 2)$$

$$P(x) = -3 + \frac{5(x^2 + 2 - 2)}{1} + \frac{3(x^2 + 2 - 2)^2}{2!}$$

$$P(x) = -3 + 5x^2 + \frac{3x^4}{2!}$$

$$P'(x) = 10x + \frac{12x^3}{2}$$

$$P'(x) = 10x + 6x^3$$

$$0 = 2x(5 + 3x^2)$$

$x = 0$ other 2 roots are complex

$$P'(x) \quad \text{---} \quad \ominus \quad \text{---} \quad \oplus \quad \text{---}$$

Because $P'(x)$ has a relative min at $x=0$, so does $g(x)$